Managing Tensions Within a Coordinated Inquiry-Based Learning Linear Algebra Course: The Role of Worksheets

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How do nine instructors teaching a linear algebra course at a research university manage tensions that emerge because of the requirement of teaching the course with an Inquiry-Based Learning approach within a coordinated system? Using Herbst's practical rationality framework (Chazan, Herbst, & Clark, 2016; Herbst & Chazan, 2011) we identify features of the course organization that contributed to tensions between professional obligations that were resolved via the production of worksheets that teachers gave to the students. We noted differences in how these tensions were handled, and provide some evidence that such differences might be related to the research orientation the instructors brought and to their status in the institution. We formulate some hypotheses that can shed light on how to assist in changing post-secondary instructional practices.

Key words: Linear algebra, Inquiry-Based Learning, Professional Obligations

Objectives

Promoting change in instructional practices in undergraduate mathematics education has been an important concern for over three decades. The calculus reform from the 80s and 90s resulted in key changes to the calculus curriculum, more prominently by bringing more contextualized and representation-rich problems that could capitalize on new hand held technologies (Douglas, 1986; Ganter, 1999; Harver, 1998). Federal support for calculus innovation through grants by the National Science Foundation increased as these were seen as a vehicle for making science, technology, engineering, and mathematics (STEM) fields more appealing to students. The need for instructional change was fueled by reports that mathematics teaching was one of the main reasons why students left STEM fields (Seymour, 1995, 2002; Seymour & Hewitt, 1997). Women in particular indicated feeling unwelcome in science and mathematics courses.

Changing instructional practices is a difficult enterprise, however, as documented by decades of research in teacher training in the K-12 system and as noted by Henderson, Beach, and Finkelstein's (2011) seminal literature review on change strategies in undergraduate STEM education. Henderson and colleagues noted that part of the problem relies on the change strategy that institutions use. They note that "developing and testing 'best practice' curricular materials and then making these materials available to other faculty" (p. 952) is ineffective in generating the anticipated change that would happen because of the availability of new materials and documents. They also noted that the second typical approach, "'top-down' policy-making meant to influence instructional practices" (p. 952) is equally ineffective as it generates arguments that usually threaten and affect collegiality so necessary for departments to function well. They note instead that the most effective strategies are aligned with or seek to change the beliefs of the individuals by involving them over a long-term process that promotes an understanding a college or university as a complex system, thus making the changes to be fully compatible with the given environment.

What happens when such approach to change is implemented in a department? This study contributes to answering that question by documenting the tensions that emerged as a department chose to implement inquiry-based learning methods (IBL) of instruction in a linear algebra course that involved about 300 students and 11 instructors. IBL is an approach that "invites students to work out ill-structured but meaningful problems… [and] construct, analyze, and critique arguments… present and discuss solutions alone at the board or via structured small-group work, while instructors guide and monitor this process" (Laursen, Hassi, Kogan, & Weston, 2014, p. 407). We followed the implementation with two goals in mind: first, to understand how the faculty operationalized teaching with inquiry-based learning methods and second, to document and understand how they managed the need to cover the prescribed content allowing at the same time free exploration of the ideas to comply with the spirit of IBL. Having information about what happens when a department seeks to institute teaching changes in a key mathematics course such as linear algebra is informative for departments interested in pursuing similar moves. This study primarily contributes to the literature on teaching change in undergraduate settings.

Theoretical framework

We assume that teaching and learning are phenomena that occur among people enacting different roles—those of teacher or students—aided by particular resources, and constrained by specific institutional requirements. We are neither concerned with the knowledge, beliefs, or attitudes of the individual teacher who enacts a particular instructional approach nor with the knowledge, beliefs, or attitudes of the students of those teachers. Rather, we seek to understand how the people in these roles navigate their obligations as teachers and students to ensure that the purposes for which they are gathered together are fulfilled.

Herbst and colleages (Chazan, Herbst, & Clark, 2016; Herbst & Chazan, 2011), using the notion of *practical rationality*, have proposed that teachers respond to four distinct professional obligations while teaching and that some of these create tensions that put teachers in a double-bind as they make decisions in the classroom. In this study we attended to the disciplinary and the institutional obligations. The disciplinary obligation refers to the expectation that teachers will represent the knowledge and practice of mathematics appropriately (e.g., precise ideas, correct language). This may include the responsibility of checking the mathematical quality of tasks, textbooks, or other resources provided to students. The institutional obligation refers to the expectation that teachers will fulfill their role as a part of larger organizations (e.g. the department, university, etc.). It involves following regimes such as official pedagogies, policies, and assessment that exist independently of teachers' individual preferences. Thus the obligation to a given course's curriculum is a manifestation of the institutional obligation, while the disciplinary obligation can "oversee[s] and question[s] the quality of the representation of the discipline offered by the curriculum" (p. 1067). We describe key tensions within the IBL linear algebra course as a clash between these two obligations precisely because of the need to teach the course using IBL.

Methods

This qualitative study took place in a coordinated linear algebra course of 11 sections taught at a research university. Two major goals for the course are stated in the syllabus as: "to learn linear algebra and to learn how to write a rigorous mathematical proof. Students should leave this course prepared to use linear algebra as well as to succeed in further theoretical courses in mathematics." The syllabus also states that the course is difficult and names two alternative courses offered to those interested in the "computational side of linear algebra."

Participants in the study were all the nine faculty members who taught the course in Fall 2015 at this university. The participants included four post-doctoral fellows, four tenured faculty, and one tier-three lecturer. Instructors were further classified as applied or non-applied mathematicians based on their background and current research interests. There were four applied (Bethany, Ed, Henry, and Miles[1]), four non-applied (Laura, Lewis, Thomas, and Ulrich), and one self-described non-applied mathematician with research interests in applied fields. Three of the instructors (Bethany, Ed, and Henry) had taught the course in the year immediately before the semester in which we collected the data and had developed enough material for the course; three instructors (Ed, Laura, and Lewis) taught the course in the semester following the data collection term (See Table 1).

Table 1: Characteristics of study participants.

Participant	Status	Research	Terms teaching Linear Algebra with IBL
Bethany	Post-doc	Applied	Winter 2015, Fall 2015
Ed	Post-doc	Applied	Winter 2015, Fall 2015, Winter 2016
Henry	Clinical faculty	Applied	Winter 2015, Fall 2015
Laura	Tenured faculty	Non applied	Fall 2015, Winter 2016
Lewis	Tenured faculty	Non applied	Fall 2015
Miles	Post-doc	Applied	Fall 2015, Winter 2016
Monica	Tenured faculty	Both	Fall 2015
Thomas	Tenured faculty	Non applied	Fall 2015
Ulrich	Post-doc	Non applied	Fall 2015

We collected several types of data: interviews with instructors, field notes from about half of the course planning meetings, observations and focus groups with students, instructor and student surveys on instructional practices, bi-weekly logs about the course from some of the faculty, and various documents: the textbook, the pacing chart, worksheets, quizzes, syllabus, and exams. We analyzed the interviews thematically by question, seeking to identify threads that were relevant to the institutional and disciplinary obligations that instructors responded to. We performed an analysis of the textbook to identify the elements that were more prominently discussed during planning meetings contrasting its content with the content present in a non-IBL textbook. We created a matrix with initial assertions, identifying the faculty to which the assertion applied to and the different sources that corroborated the assertion, in this way engaging in a cross-case analysis and triangulation of the data sources.

In order to corroborate our interpretations, we sent a summary of our findings to our participants so that they could verify or rectify what we have found. Three of eight participants (Bethany, Henry, and Laura) responded making specific suggestions about the textbook analysis (part of a longer version of this paper) and corrected some factual information regarding the context of the course. Given their comments and who responded we have reason to believe that our interpretations accurately reflect the situation that we are about to describe.

There are two limitations to our study. First, we are still in the process of collecting student performance data. As of now, we have data from focus groups on student perceptions of how IBL supported their learning, but we do not have access to student final grades in the course or in subsequent courses. Thus, although we could talk about impact of the change we are in no position to do so. Additionally, this study seeks only to document processes of change not the impact of the change on students. The students were quite satisfied with the learning they were experiencing, but self-report data is hardly sufficient as a tool to convince administration of the appropriateness of a programmatic instructional change. In a subsequent phase we plan to perform a historical analysis of data to study impact of the implementation

[1] Names are pseudonyms.

of IBL. Second, the first and third authors were involved as evaluators in a training grant that involved the participating faculty. The goal of the grant was to increase the number of undergraduate students in the lower division courses experiencing IBL. As evaluators, their role was to document how the IBL training strategies were deployed and to assess which ones were perceived by the faculty as beneficial in helping them understand how to teach with IBL. As such, some of the data collected sought to providing feedback to faculty rather than provide data for our research questions. For that reason, data related to faculty feedback (observations and focus groups) play a secondary, albeit important role, that of triangulation, in this analysis.

Results

The department had put in place a number of mechanisms to ensure that all the students in the 11 sections of the course were exposed to similar content and experiences. These mechanisms came in the form of the pacing chart, that indicated the sections of the textbook to cover each day; the textbook itself that students had to use to do their daily reading quiz and that instructors should use to design worksheets; the homework, which was the same for all students in the course; and the exams (two midterms and a final) which were also common and graded with a common rubric. Two tensions emerged from the need to fulfill the obligations while adhering to those mechanisms._The first tension expressed itself through complaints regarding the presentation of content in the textbook required by the department that differed from what some faculty thought should be the definitions that should be used, an evidence of their allegiance, or obligation towards upholding the knowledge in the discipline. This tension is especially interesting to observe in an IBL setting because instructors distribute supplementary material in the form of worksheets where they can choose to support or deviate from the textbook.

The second tension emerged from the two explicitly stated goals of the course, "to learn linear algebra and to learn how to write a rigorous mathematical proof" and what instructors saw was the best way to approach the linear algebra content, either by exemplifying notions through concrete examples that then get formalized or by starting with definitions that are then exemplified with concrete examples. We saw the faculty resolving this tension in their worksheet production by either emphasizing applications of concepts or by increasing the level of abstraction and development of mathematical theory. We believe that their choices were related to their research interests. We expand on the first tension here.

According to Henry, the selection of Brestcher as the textbook for the course was predicated on the desire to choose a book that helped develop intuition. This was the book he felt students could best learn from. Henry chose the textbook after reviewing various options and, after having given it "a try," found it sufficiently readable and usable by the students. His perception was that the problems were "decent" and sufficient for helping students develop the needed intuitive understanding of linear algebra ideas, despite its shortcomings regarding the definitions used for linear independence and linear transformations, and the particular ordering of the chapters. During the first semester in which all the sections used IBL with Bretscher the faculty heatedly discussed these shortcomings during the weekly course planning meetings.

Confirming those discussions, the interviews with faculty suggested that nearly all instructors (7 out of 9) recognized problems with the mathematics in the required textbook (e.g., "This book is almost like not having a book. It has good problems and historical treatment, but doesn't have the definitions that mathematicians use," Thomas). Instructors with applied backgrounds handled this problem differently than instructors with non-applied background. The former group sought to maintain consistency of content and pacing across the different sections of the course, making an effort to align the content with the required readings. For example, Bethany, a professor with an applied background, said,

> A lot of it came down to how much we should rely on the book versus how much you should rely on what you think is convention. From my perspective, this is the book that students read, and we should try our best to provide them a consistent picture. I'll sacrifice nonstandard definitions for the sake of that consistency.

Similarly, Ed said that once the textbook is chosen, he sticks to it even if it is not ideal. He figured he could be creative with the worksheets and be more rigorous if he wanted to. He also said that Bretscher's definitions (e.g., linear transformations) were good for conceptual understanding but not for theory. Henry, who choose the textbook for the course, was also an applied mathematician. In contrast, faculty with a non-applied background prioritized mathematical development. Lewis did not require students to read the book because he was unhappy with the sequence of topics: "The book was awful on many counts, even their definitions were suspect." Monica, while unhappy with the textbook, did not deviate from it. She said, "Certain things were in the wrong order," for example, the way the book introduced vector spaces. She liked the use of examples to develop students' intuition, "...but then the book stays with the example of $\mathbf{R}^n$ and if that's the first time you hear about a vector space, which is really an abstract concept, …it does harm [if] that's all we know." Like Ed, she found the worksheets to be a place that she could supplement the book. She tried to build an abstract conception of vector spaces and general structures in class and in the worksheets and also through lectures in parallel to the textbook's examples in $\mathbf{R}^n$.

Henry and Lewis wrote the original worksheets that were used by most of the instructors: Bethany, Ed, and Miles used and revised Henry's worksheets while Thomas and Laura used or referred to Lewis'. We noted that the first group includes all instructors with applied background, while the second group includes faculty with non-applied background. Henry's worksheet (Figure 1) from the section on subspaces of $\mathbf{R}^n$, bases, and linear independence (section 3.2) contains exercises that use of the term *redundant*, a term introduced by Bretscher (2013, p.125) to create an intuitive build-up to linear independence. However, the worksheet also contains a problem that walks students through a proof using the standard definition of linear independence.[2] Lewis's worksheet (Figure 2) on linear transformations (sections 2.1 and 2.2) begins, "Our definition will be a bit different than the one in the book but you will see that it is equivalent to the definition in the book." His worksheet on linear independence, bases, spanning sets, and basis (section 3.2) begins with the standard definition of *linear independence*, *span*, and a *basis*, and briefly mentions *redundant* in his "Something to think about" section at the end of the worksheet. Thus, Henry started with the book's definition and related it to the standard definition, while Lewis started with the standard definitions and related them back to the textbook.

[2] A subset S of a vector space V is called **linearly dependent** if there exists a finite number of distinct vectors $u_1, \dots, u_n$ in S and scalars $a_1, \dots, a_n$, not all zero, such that $a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$. In this case we also say that the vectors of S are linearly dependent. A subset S of a vector space V that is not linearly dependent is called **linearly independent**. As before, we also say that the vectors of S are linearly independent (Friedberg, Insel, & Spence, 2002, p. 36-37).

3. More fun with linear independence!

   (a) Are the vectors $\vec{v}_1 = \begin{bmatrix}1 & 0 & 0 & -2\end{bmatrix}^T$ and $\vec{v}_2 = \begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}^T$ linearly independent? Why or why not?

   (b) Now, consider the five vectors $\vec{v}_1 = \begin{bmatrix}1 & 0 & 0 & -2\end{bmatrix}^T$ and $\vec{v}_2 = \begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}^T$, $\vec{v}_3 = \begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}^T$, $\vec{v}_4 = \begin{bmatrix}-2 & 2 & 3 & 4\end{bmatrix}^T$ and $\vec{v}_5 = \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^T$. Which is the first redundant vector? Write it as a linear combination of the preceding vectors.

   (c) Write a nontrivial linear relation among $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$, and $\vec{v}_5$. *(Hint: your work in (b) has the outside chance of being useful.)*

4. Wasn't that fun? What fun thing should we do next? Let's suppose we have a set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$.

   (a) If $m = 5$ and we know that we have the linear relation

   $$-\vec{v}_1 + 2\,\vec{v}_2 + 3\,\vec{v}_3 - \vec{v}_4 + 0\,\vec{v}_5 = 0,$$

   explain why $\{\vec{v}_1, \ldots, \vec{v}_5\}$ are linearly dependent (by identifying a redundant vector in the set).

   (b) Explain why the existence of a nontrivial linear relation $c_1\vec{v}_1 + \ldots + c_m\vec{v}_m = 0$ (nontrivial means that at least one $c_i \neq 0$), shows that there is a redundant vector, and hence that $\{\vec{v}_1, \ldots, \vec{v}_m\}$ are linearly dependent.

   (c) Conversely, if $\vec{v}_i$ is a redundant vector in the set $\{\vec{v}_1, \ldots, \vec{v}_m\}$, show that there is a nontrivial relation among the vectors in this set.

Figure 1. *Sample of Henry's worksheet from the section on subspaces.*

A *basis* of V is a collection of elements that is both *linearly independent* and a *spanning set.*

1. Decide if the following collection of vectors are a basis for $\mathbf{R}^3$. If not, say what property or properties fail.

   (a)
   $$\vec{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

   (b)
   $$\vec{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}2\\1\\3\end{bmatrix}.$$

   (c)
   $$\vec{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}2\\1\\2\end{bmatrix}.$$

   (d)
   $$\vec{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}2\\1\\2\end{bmatrix}, \vec{v}_4 = \begin{bmatrix}0\\1\\-1\end{bmatrix}$$

2. Let V be the set of vectors $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ in $\mathbf{R}^3$ satisfying the relation

   $$x_1 + 2x_2 - 4x_3 = 0.$$

   (a) Prove that V is a subspace of $\mathbf{R}^3$.

   (b) Find a basis for V.

Something to think about

Vector $\vec{v}$ in Bretscher's book was overheard saying "I hate being a redundant vector. If I was the zero vector, that would be okay. But if I'm redundant just because others were ahead of me in the line, then that's not fair!" Do you sympathize with vector $\vec{v}$?

Figure 2. *Sample of Lewis' worksheet from the section on subspaces.*

These examples illustrate the tension between the obligation to comply with the institutional requirement of using a textbook that will be consistently used across all sections of the course, and the disciplinary obligation to represent accurate mathematics when the goal is to foster proving and illustrate how instructors managed them through the writing of the worksheets, suggesting a possible explanation rooted on the research background of the instructors. This is speculative, as other factors (e.g., institutional status) might have played a role as well.

Discussion and Implications

The instructors were able to use the worksheets as a place to manage the tensions that arose from their institutional and disciplinary obligations. The instructors who were unhappy with the content of the textbook were able to provide supplementary material to their students that, in their opinion, displayed the mathematics more appropriately. In particular, we found instances where instructors preferred different yet equivalent definitions, and used the worksheets to connect them to the curriculum.

All the instructors expressed deep commitments to their students, which indicates that these decisions reflected their interest in student success. Research on how courses are designed can help the research community build knowledge on what speaks to faculty teaching students in STEM environments. The preliminary evidence from this investigation suggests that the tensions that can emerge as mathematics departments seek to increase the number of students experiencing new ways of teaching can be navigated via a coordination system that allows faculty some space for exerting their professional commitments.

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