Emerging Insights from the Evolving Framework of Structural Abstraction in Knowing and Learning Advanced Mathematics

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Only recently 'abstraction on objects' has attracted attention in the literature as a form of abstraction that has the potential to take account of the complexity of students' knowing and learning processes compatible with their strategy of giving meaning. This paper draws attention to several emerging insights from the evolving framework of structural abstraction in students' knowing and learning of the limit concept of a sequence. Particular ideas are accentuated that we need to understand from a theoretical point of view since they reveal a new way of understanding knowing and learning advanced mathematical concepts and have significant implications for educational practice.

Keywords: Limit Concept; Mathematical Cognition; Sense-Making; Structural Abstraction

Introduction

Theoretical and empirical research shows the existence of differences in knowing and learning concerning different kinds of knowledge (diSessa, 2002). A general framework on abstraction cannot encompass the whole complexity of knowing and learning processes in mathematics. Rather, in investigating the nature, form, and emergence of knowledge pieces, various micro-genetic learning theories may be developed, which will be quite specific to particular mathematical concepts, individuals, and their underlying sense-making strategies. As a consequence, the complexity of knowing and learning processes in mathematics cannot be described or explained by only one framework. Instead, we acknowledge that comprehensive understanding of cognition and learning in mathematics draws on a variety of theoretical frameworks on abstraction.

The literature demonstrates significant theoretical and empirical advancement in understanding 'abstraction-from-actions' approaches, particularly the cognitive processes of forming a (structural) concept from an (operational) process (Dubinsky, 1991; Gray & Tall, 1994; Sfard, 1991). Abstraction-from-actions approaches take account of a certain sense-making strategy, namely what Pinto (1998) described as 'extracting meaning'. However, only recently 'abstraction on objects' has attracted attention as a form of abstraction that provides a new way of seeing the complex knowing and learning processes compatible with students' strategy of what Pinto (1998) described as 'giving meaning'.

The purpose of this paper is to provide deeper meaning to a recently evolving framework of a particular kind of 'abstraction from objects': structural abstraction. The structural abstraction framework is evolving in the sense that the framework functions both as a tool for research and as an object of research (Scheiner & Pinto, submitted). In more detail, we use the structural abstraction framework retrospectively as a lens through which we reinterpret a set of findings on students' knowing and learning of the limit concept of a sequence. This reinterpretation is an active one in the sense that the framework serves as a tool to analyze a set of data, while the framework is also refined and extended since the reinterpretation produces deeper insights about the framework itself. Especially, these more profound insights are what we need to understand from a theoretical point of view since they have relevance for significant issues in knowing and learning advanced mathematical concepts.

We begin by providing an upshot of our synthesis of the literature on abstraction in knowing and learning mathematics. Our synthesis is to suggest an orientation toward the evolving framework of structural abstraction as an avenue to take account of an important area for consideration – that is, drawing attention to the complex knowing and learning processes compatible with students' 'giving meaning' strategy. The structural abstraction framework constitutes the foundation of the second part of the paper providing emerging insights in knowing and learning the limit concept of a sequence. These insights offer theoretical advancement of the framework and deepen our understanding of knowing and learning advanced mathematics.

Mapping the Terrain of Research on Abstraction in Mathematics Education

Abstraction seems to have gained a bad reputation because of the criticism articulated by the situated cognition (or situated learning) paradigm, and, as a consequence, has almost disappeared. This criticism rests primarily on traditional approaches considering abstraction as decontextualization and often confusing abstraction with generalization. The recent contribution by Fuchs et al. (2003) shows that such classical approaches to abstraction still exist:

> "To abstract a principle is to identify a generic quality or pattern across instances of the principle. In formulating an abstraction, an individual deletes details across exemplars, which are irrelevant to the abstract category […]. These abstractions […] avoid contextual specificity so they can be applied to other instances or across situations." (p. 294)

However, scholars in mathematics education argued against the decontextualization view of abstraction. Van Oers (1998, 2001), for instance, argued that removing context must impoverish a concept rather than enrich it. Several other scholars have reconsidered and advanced our understanding of abstraction in ways that account for the situated nature of knowing and learning in mathematics. Noss and Hoyles (1996) introduced the notion of *situated abstraction* to describe "how learners construct mathematical ideas by drawing on the webbing of a particular setting which, in turn, shapes the way the ideas are expressed" (p. 122). Webbing in this sense means "the presence of a structure that learners can draw up and reconstruct for support – in ways that they can choose as appropriate for their struggle to construct meaning for some mathematics (Noss & Hoyles, 1996, p. 108). Hershkowitz, Schwarz, and Dreyfus (2001) introduced the notion of *abstraction in context* that they presented as "an activity of vertically reorganizing previously constructed mathematics into new mathematical structure" (p. 202). They identify abstraction in context with what Treffers (1987) described as 'vertical mathematization' and propose entire mathematical activity as the unity of analysis. These contributions demonstrate that research on abstraction in knowing and learning mathematics has made significant progress in taking account of the context-sensitivity of knowledge.

Several other contributions shape the territory in mathematics education research on abstraction. Scheiner (2016) proposed a distinction between two forms of abstraction, namely *abstraction on actions* and *abstraction on objects*. This distinction has been further refined in Scheiner and Pinto (2014) arguing that the focus of attention of each form of abstraction takes place on physical objects (referring to the real world) or mental objects (referring to the thought world) (see Fig. 1).

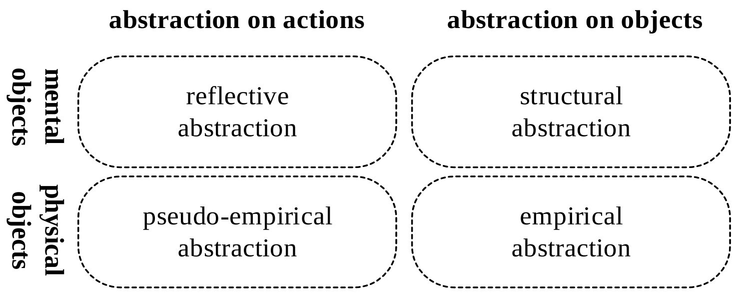

Fig. 1: A frame to capture various kinds of abstraction (reproduced from Scheiner & Pinto, 2014)

We consider this distinction as being productive in trying to capture some of the variety of images of abstraction in mathematics education (for details see Scheiner & Pinto, 2016). It acknowledges Piaget's (1977/2001) three kinds of abstraction, including pseudo-empirical abstraction, empirical abstraction, and reflective abstraction, that served as critical points of departure in thinking about abstraction in learning mathematics. Research on abstraction in mathematics has long moved beyond classifying and categorizing approaches in cognition and learning. For instance, Mitchelmore and White (2007), in going beyond Piaget's empirical abstraction and in drawing on Skemp's (1986) conception of abstraction, described abstraction in learning elementary mathematics concerning seeing the underlying structure rather than the superficial characteristics. Abstraction in learning advanced mathematics, however, is almost always defined in terms of *encapsulation* (or *reification*) of processes into objects, originating in Piaget's (1977/2001) idea of *reflective abstraction*. Reflective abstraction is an abstraction from the subject's actions on objects, particularly from the coordination between these actions. The particular function of reflective abstraction is abstracting properties of an individual's action coordination. That is, reflective abstraction is a mechanism for the isolation of specific properties of a mathematical structure that allows the individual to construct new pieces of knowledge. Taking Piaget's reflective abstraction as a point of departure, Dubinsky and his colleagues (Dubinsky, 1991; Cottrill et al., 1996) developed the APOS theory describing the construction of concepts through the encapsulation of processes. Similar to encapsulation is reification – the central tenet of Sfard's (1991) framework emphasizing the cognitive process of forming a (structural) concept from an (operational) process. In the same way, Gray and Tall (1994) described this issue as an overall progression from procedural thinking to proceptual thinking, whereas proceptual thinking means the ability to flexibly manipulate a mathematical symbol as both a process and a concept. Gray and Tall (1994) termed symbols that may be regarded as being a pivot between a process to compute or manipulate and a concept that may be thought of as a manipulable entity as procepts.

Scheiner (2016) revealed that the literature shows an unyielding bias toward *abstraction on actions* as the driving form of abstraction in knowing and learning advanced mathematics. This almost always exclusive view arises directly from the trajectory of our field's history; originating in Piaget's assumption that only reflective abstraction can be the source of any genuinely new construction of knowledge. While abstraction-on-actions approaches have served many purposes quite well, they cannot track detail of students' knowing and learning processes compatible with the strategy of giving meaning. The recently evolving framework of structural abstraction has attracted attention as a promising tool to shed light into the complexity of students' knowing and learning processes compatible with their strategy of 'giving meaning'.

The Evolving Framework of Structural Abstraction

The evolving framework of structural abstraction (Scheiner, 2016) further elaborates Tall's (2013) conception of this particular kind of abstraction. The crucial aspect lies in the argument that structural abstraction takes account of two processes: (1) complementizing meaningful aspects and structure underlying specific objects falling under a particular concept, and (2) promoting the growth of a complex knowledge system through restructuring various knowledge pieces. Several assumptions guide the evolving framework of structural abstraction assumptions:

Concretizing through Contextualizing

Structural abstraction takes place on mental objects that, in Frege's (1892a) sense, fall under a particular concept. These objects may be either concrete or abstract. Concreteness and abstractness, however, are not considered as properties of an object but rather as properties of an individual's relatedness to an object in the sense of the richness of a person's representations, interactions, and connections with the object (Wilensky, 1991). From this point of view, rather than moving from the concrete to the abstract, individuals, in fact, begin their understanding of (advanced) mathematical concepts with the abstract (Davydov, 1972/1990). The ascending from the abstract to the concrete requires a *concretizing* process where the mathematical structure is particularized by looking at the object in relation with itself or with other objects that fall under the particular concept. The crucial aspect for concretizing is *contextualizing*, that is, setting the object(s) in different specific contexts. Different contexts may provide various senses (Frege, 1892b) of the concept of observation.

Complementizing through Recontextualizing

The central characteristic of the structural abstraction framework is that while, within the empiricist view, conceptual unity relies on the commonality of elements, it is the interrelatedness of diverse elements that creates unity within the approach of structural abstraction. The process of placing objects into different specific contexts allows particularizing essential components. Structural abstraction, then, means attributing the particularized meaningful components of objects to the mathematical concept. Thus, the core of structural abstraction is *complementarity* rather than similarity. The meaning of advanced mathematical concepts is developed by complementizing diverse meaningful components of a variety of specific objects that have been contextualized and recontextualized in multiple situations. This perspective agrees with van Oers' (1998) view on abstraction as related to *re*contextualization instead of *de*contextualization.

Complexifying through Complementizing

The structural abstraction framework takes the view that knowledge is a complex system of many kinds of knowledge elements and structures. Complementizing implies a process of restructuring the system of knowledge pieces. These knowledge pieces have been constructed through the above-mentioned process or are already constructed elements coming from other concept images, which are essential for the new concept construction. The cognitive function of structural abstraction is to facilitate the assembly of more complex and compressed knowledge structures. Taking this perspective, we construe structural abstraction as moving *from simple to complex* knowledge structures, a movement with the aim to build coherence among the various knowledge pieces through restructuring them.

Emerging Insights from the Structural Abstraction Framework

In this section, we summarize emerging insights we gained, and still gain, by using the evolving framework of structural abstraction retrospectively as a lens through which findings on students' (re-)construction of the limit concept of a sequence were reinterpreted. The study by Pinto (1998) provides the context in which she identified mathematics undergraduates' sense-making strategies of formal mathematics. From a cross-sectional analysis of three pairs of students, two prototypical strategies of making sense could be identified, namely 'extracting meaning' and 'giving meaning':

> "*Extracting meaning* involves working within the content, routinizing it, using it, and building its meaning as a formal construct. *Giving meaning* means taking one's personal concept imagery as a starting point to build new knowledge." (Pinto, 1998, pp. 298-299)

The literature on abstraction-from-actions provides several accounts of how students construct a mathematical concept compatible with their strategy of 'extracting meaning'; however, there are almost no accounts of how students construct a concept compatible with their strategy of 'giving meaning'. It is important to note that the evolving framework of structural abstraction is problem driven, that is, addressing the need of bringing light into the complexity of students' knowing and learning processes compatible with their strategy of 'giving meaning', rather than filling a theoretical gap just because it exists. The reinterpretation of empirical data on students' strategies of giving meaning in the light of the theoretical framework of structural abstraction proved to be particular fruitful - not only to provide deeper insights into the strategy of giving meaning but also as a way to deepen our understanding of the phenomenon of structural abstraction that revealed to new theoretical development (Pinto & Scheiner, 2016; Scheiner & Pinto, 2014). In the following pages, we highlight the main theoretical advancements.

The idea of complementizing meaningful components underlying the structural abstraction framework reflects the idea that whether an individual has 'grasped' the meaning of a mathematical concept depends on the specific context where the objects falling under the particular mathematical concept have been placed in. This supports Skemp's (1986) viewpoint that "the subjective nature of understanding […] is not […] an all-or-nothing state" (p. 43). The reanalysis of the data indicates that the object of researchers' observation should be directed to students' *partial constructions* of the limit concept. These partial constructions may be specific and productive to particular contexts but may remain not fully connected and may be unproductive in other contexts (for instance, in making sense of the formal definition). The empirical data shows that, in the case of the students who give meaning, several meaningful elements and relations in understanding the limit concept of a sequence are involved, although a few elements are missing (or distorted). However, some students are able to (re-)construct some meaningful components at need by making use of their partial constructions, while others are not able to do so.

The reanalysis indicates that some students have developed *resources* that enable them to (re-)construct the limit concept of a sequence at need. Scheiner and Pinto (2014) presented a case where a student developed a generic representation of the limit concept of a sequence that operates well in several, although not all, contexts and situations. This particular representation, however, allows the student to (re-)construct the limit concept in other contexts and situations. The reinterpretation of the data sheds light on the phenomenon that individuals may do not 'have' all relevant meaningful components, but, rather, they may have resources to generate some meaningful components and make sense of the context at need. In that sense, the 'completeness' of the complementizing process cannot ever be taken as absolute.

Several scholars suggested exposing learners to multiple contexts and situations. An important insight from using the structural abstraction framework retrospectively is that exposure to multiple contexts is at least important for particularizing meaningful components: various objects falling under a particular mathematical concept have to be set into different specific contexts in order to make visible the meaningful components or mathematical structure of these objects. In so doing, the objects may be ‘exemplified’ through a variety of representations, in which each representation has the same reference (the mathematical object); however, different representations may express different senses depending on the selected representation system (see Fig. 2). The distinction between *sense* and *reference* has been specified by Frege (1892b) in his work *Über Sinn und Bedeutung*, indicating both the sense and the reference as semantic functions of an expression (a name, sign, or description). In short, the former is the *way* that an expression refers to an object, whereas the latter is the *object* to which the expression refers. According to Frege (1892b), to each representation correspondents a sense; the latter may be connected with an *idea* that can differ within individuals since people might associate different senses with a given representation. Though multiple contexts and situations are needed, a new context that does not provide a new sense will unlikely to be productive for the concept construction. The framework indicates that the *nature of the contexts* the objects are set is determinative of their value toward the complementizing process.

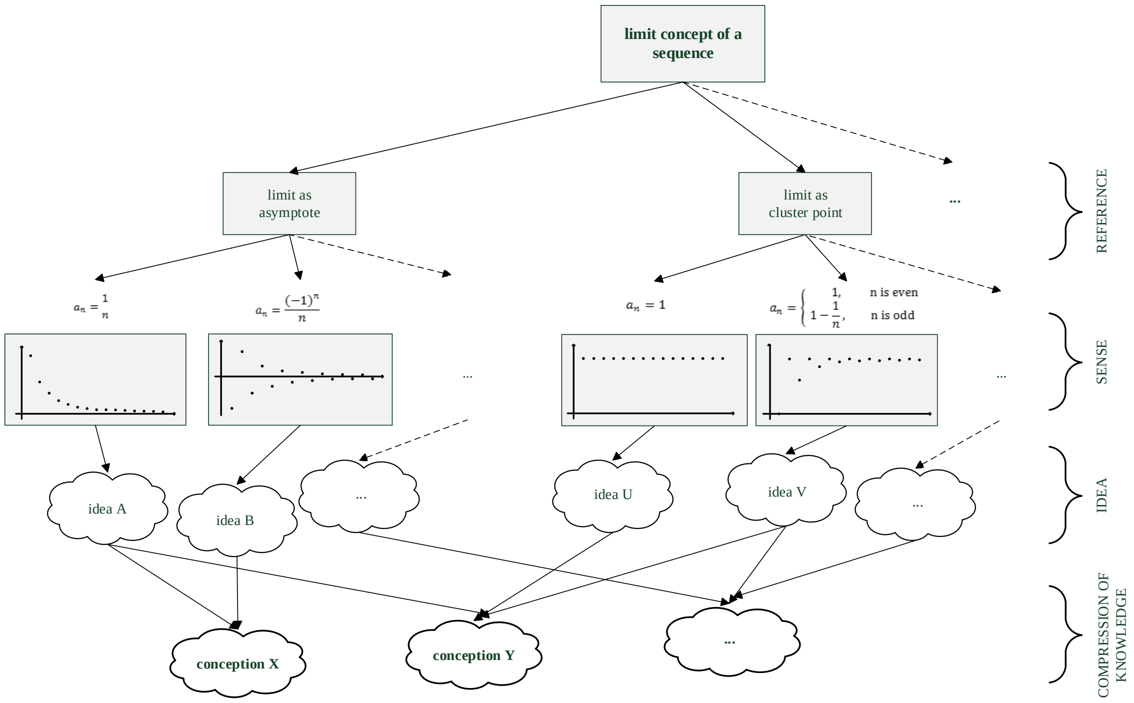

Fig. 2: Reference, sense, and idea

Research also indicates that students may have difficulties with the *relationships* between the sense and the reference as well as difficulties in *maintaining* the reference as the sense changes (Duval, 1995, 2006). Thus, based on the insights we have gained from reanalyzing the data (Pinto & Scheiner, 2016; Scheiner & Pinto, 2014), we can assume that these difficulties may (at least partly) be overcome by providing students a particular resource (such as a generic representation of the mathematical concept) that serves as a guiding tool in complementizing the meaningful components indicated in the different senses. From this perspective, a ‘representation for’ is a tool that provides theoretical structure in constructing the meaning of the concept of observation. It necessarily reflects essential aspects of a mathematical concept but can have different manifestations (Van den Heuvel-Panhuizen, 2003). Concerning the learning of the limit concept of a sequence, the reinterpretation of the data indicates that a slightly modified version of a student's representation (see Fig. 3) may support the complementizing process when the limit concept is recontextualized.

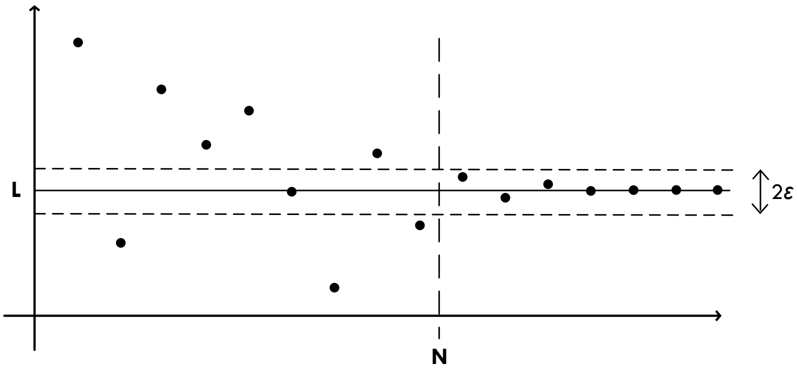

Fig. 3: A generic representation for learning the limit concept of a sequence

Notice that this generic representation for learning the limit concept of a sequence takes account of several students' common conceptions identified in the research literature, including those as (1) the limit is unreachable, (2) the limit has to be approached monotonically, and (3) the limit is a bound that cannot be crossed (see Cornu, 1991; Davis & Vinner, 1986; Przenioslo, 2004; Tall & Vinner, 1981; Williams, 1991).

The reanalysis of the empirical data gained from Pinto's (1998) study has shown that students giving meaning built a representation of the concept and, at the same time, used it as a representation for recognizing and building up knowledge – the reconstruction of the formal concept definition, for instance. The analysis shows that these students consistently used *representations of* mathematical objects to create pieces of knowledge; or, in other words, the representations were actively taken as *representations for* emerging new knowledge and making sense of the context and situation. This shift from constituting a representation of the limit concept to using this representation as a representation for (re-)constructing the limit concept in other contexts can be described in terms of shifting from a *model of* to a *model for* (Streefland, 1985) – a shift from an *after*-image of a piece of given reality to a *pre*-image for a piece of to be created reality. This mental shift from 'after-image' to 'pre-image' indicates a degree of awareness of the meaningful components and the complexity of knowledge structure that allows the transition from a 'representation of' as a result of various representations expressing specific objects set in different contexts to a 'representation for' constructing and reconstructing the limit concept, inter alia, in formal mathematical reasoning. We suggest that a generic representation, as presented in Fig. 3, may provide an instructional tool that supports raising the awareness of meaningful components in learning the limit concept of a sequence. In other words, such a generic representation may direct students' perception of meaningful components.

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