Student Mathematical Connections in an Introductory Linear Algebra Course

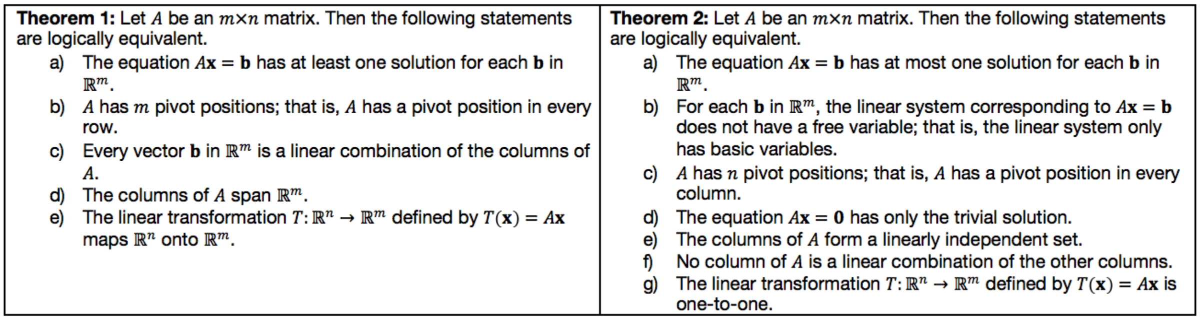
Spencer Payton
Washington State University

In an introductory linear algebra course, students are expected to learn a plethora of new concepts as well as how these concepts are connected to one another. Learning these connections can be quite challenging for students due to the vast number of connections and student inexperience with mathematical logic. The study reported here consisted of an investigation into how inquiry-oriented teaching methods could be employed in an attempt to create opportunities for students to develop mathematical connections in an introductory linear algebra course.

Key words: linear algebra, mathematical connections, inquiry-oriented teaching

Introductory linear algebra courses have traditionally been quite challenging for students. There are several reasons for this, including the fact that students are introduced to a plethora of brand new concepts and terminology. Further, many of these concepts are connected to one another in various ways, and students are expected to learn these connections as well. While many researchers and teachers would agree that students should be able to make mathematical connections, the phrase "mathematical connection" is often loosely defined. This study considers one particular type of mathematical connection in an introductory linear algebra course: *logical implication connections*. Relationships between various linear algebraic concepts are often summarized in theorems of logical equivalence such as the Invertible Matrix Theorem (IMT) (Lay, 2011). The statements in this theorem are all logically equivalent, meaning any statement in the theorem logically implies another (and vice versa). Thus, the logical implications present in the IMT could be described as logical implication connections.

While the IMT provides a convenient presentation of logical implications in introductory linear algebra, it is somewhat restrictive due to the fact that it only applies to square coefficient matrices (as only square matrices can be invertible). However, subsets of the logical implications inherent in the IMT could be applied to non-square matrices. The IMT could actually be divided into two "sub-theorems," which will hereby be known as the First and Second Theorems of Logical Equivalence; these theorems are presented in Figure 1.



Theorem 1: Let A be an $m \times n$ matrix. Then the following statements are logically equivalent.

a) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^m$.
b) A has m pivot positions; that is, A has a pivot position in every row.
c) Every vector $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
d) The columns of A span $\mathbb{R}^m$.
e) The linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

Theorem 2: Let A be an $m \times n$ matrix. Then the following statements are logically equivalent.

a) The equation $A\mathbf{x} = \mathbf{b}$ has at most one solution for each $\mathbf{b}$ in $\mathbb{R}^m$.
b) For each $\mathbf{b}$ in $\mathbb{R}^m$, the linear system corresponding to $A\mathbf{x} = \mathbf{b}$ does not have a free variable; that is, the linear system only has basic variables.
c) A has n pivot positions; that is, A has a pivot position in every column.
d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e) The columns of A form a linearly independent set.
f) No column of A is a linear combination of the other columns.
g) The linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.

Figure **1**: Unlike the Invertible Matrix Theorem, these theorems of logical equivalence are not restricted only to the case of square matrices.

As learning these mathematical connections can be challenging for students, it would be beneficial to improve the teaching of these connections. The study described in this report was part of a larger study that attempted to determine how inquiry-oriented teaching methods could be implemented in an introductory linear algebra course that, due to considerations such as large class size and limited amount of class time, would not lend itself to the traditional demands of inquiry-oriented teaching. Regarding the teaching of mathematical connections, one goal of this study was to answer the following research question: How do students take advantage of inquiry-oriented teaching to make connections in an introductory linear algebra class?

Literature Review

It is not uncommon for students to know that two linear algebraic concepts are connected but not understand *why* they are connected. This issue was well described by Harel:

> So if a student thinks of 'linear independence' to mean 'the echelon matrix which results from elimination has no rows of zeros,' without being able to mathematically justify this connection, then he or she does not understand the concept of linear independence. (Harel, 1997, p. 111)

This issue of the quality of student understanding has been previously discussed by Skemp (1987) in his description of instrumental and relational understanding. The understanding presented in Harel's example is *instrumental understanding*; that is, knowing what to do but not why. According to Skemp, true understanding of a concept involves *relational understanding*, which is "knowing both what to do and why" (Skemp, 1987, p. 153). This characterization of the quality of understanding could be applied to mathematical connections. Thus, a student has made an *instrumental connection* if the student has formed a connection but does not understand why that connection exists; similarly, a student has made a *relational connection* if the student has formed a connection and understands why that connection exists. For example, a student could present relational understanding of a logical implication connection if he or she can form a chain of logical implications beginning with one statement in a theorem of logical equivalence and ending with another. Unfortunately, students at this level often struggle with mathematical logic, and in particular, many students struggle to form these chains of reasoning (Dorier & Sierpinska, 2001).

Regarding inquiry-oriented teaching, there are several ways to define inquiry depending on the context or academic subject. Rasmussen and Kwon (2007) characterize student inquiry in a mathematics class through Richards' (1991) definition of mathematical inquiry, which is the mathematics of mathematically literate adults. Thus, mathematical inquiry involves participating in mathematical discussion, solving new problems, listening to mathematical arguments, and proposing conjectures. With this interpretation of mathematical inquiry, inquiry-oriented teaching involves creating opportunities for students to engage in mathematical inquiry.

Setting and Methods of Analysis

This study was conducted through an action research methodology that began with a pilot study in the summer of 2015 and continued into the fall of 2015 and spring of 2016. Each of these action research cycles consisted of research in an introductory linear algebra course that was taught by the researcher. While the results of the pilot study and fall 2015 research cycle informed the spring 2016 research cycle, this report will primarily focus on the spring 2016 cycle.

In the spring of 2016, the researcher taught an introductory linear algebra course at a large state university in the Pacific Northwest. The class consisted of sixty students; the majority of these students were engineering majors, while others were mainly math and computer science majors. The course was a two credit course, which placed considerable time constraints on the instructor. As a result of these constraints, inquiry-oriented teaching activities were largely reserved for concepts closely related to logical implication connections. In particular, students worked on several activities focusing on span and linear independence, as these concepts play parallel roles in the first two theorems of logical equivalence; two of these activities are presented in Figure 2.

Without solving a linear system or using any elementary row operations, determine whether the following sets of vectors span the given space. For each set of vectors, formulate a conjecture about span based on that set. • Do the vectors $\begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}-2\\4\end{bmatrix}$ span all of $\mathbb{R}^2$? • Do the vectors $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}$ span all of $\mathbb{R}^3$? • Do the vectors $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}$ span all of $\mathbb{R}^3$? • Do the vectors $\begin{bmatrix}1\\7\\0\end{bmatrix}, \begin{bmatrix}3\\8\\0\end{bmatrix}, \begin{bmatrix}-2\\9\\0\end{bmatrix}$ span all of $\mathbb{R}^3$? • Do the vectors $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}$ span all of $\mathbb{R}^2$?	Without solving a linear system or using any elementary row operations, determine whether the following sets of vectors are linearly independent or dependent. For each set of vectors, formulate a conjecture about linear independence or dependence based on that set. • $\left\{\begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}-2\\4\end{bmatrix}\right\}$ • $\left\{\begin{bmatrix}1\\3\\-5\end{bmatrix}, \begin{bmatrix}3\\9\\-10\end{bmatrix}\right\}$ • $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-3\\-4\end{bmatrix}, \begin{bmatrix}-2\\4\end{bmatrix}\right\}$ • $\left\{\begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\8\end{bmatrix}\right\}$

Figure **2**: Each of these activities were designed as opportunities for students for explore span and linear independence in ways that would allow them to develop logical implication connections involving span and linear independence.

Data on student mathematical connections was largely collected from interviews with nine students from the aforementioned class. These interviews were conducted shortly after the Invertible Matrix Theorem had been covered in class. Each interview was approximately an hour in length and consisted of students finding the solution set of a linear system, a vector equation, and a matrix equation. Each problem lent itself to a different theorem of logical equivalence. For example, the coefficient matrix corresponding to the linear system had a pivot position in every row, thus making every statement from Theorem 1 true for that coefficient matrix. Similarly, the vector equation lent itself to Theorem 2 and the matrix equation lent itself to the IMT. After an interviewee completed one of the problems, the interviewee was asked to describe his or her work. The researcher would then present the interviewee with a list of vocabulary terms that had been discussed in class. The interviewees were asked to discuss as many of the vocabulary terms as they could and how they relate to each problem. The interviewer would often ask for justification of particular claims that the interviewee had made and would sometimes directly ask the interviewee whether he or she could discuss a particular vocabulary term. This was all done in an attempt to determine what logical implication connections the interviewees could evoke that incorporated some of the familiar terms involved in the theorems of logical equivalence.

In analyzing the interviews, the researcher attempted to determine what mathematically correct logical implications corresponding to the three theorems of logical equivalence each interviewee evoked. Evidence of logical implication connections took several forms. Many logical implications involved words such as if, then, means, because, and so. For example, “The vectors, if a linear combination of those produce every single vector in that space, then they span that space” would be considered a logical implication connection. While many logical implications were evoked entirely by the interviewees, some logical implications were evoked as a result of an interviewee responding to a question asked by the interviewer. After determining what logical implications the interviewees evoked, the researcher then attempted to determine which of these connections were relational connections; this was largely accomplished by determining which logical implication connections a student was able to justify.

Results

In general, the interviewees tended to evoke more connections relevant to the second theorem of logical equivalence than they did the first. This is in itself not entirely surprising; the theorems presented in Figure 1 were the versions of the theorem discussed in class, and the second theorem contains more statements than the first. As the concept of invertibility and the IMT were still new to the interviewees, they tended to evoke relatively few connections exclusive to the IMT. Due to this, the results reported here will primarily focus on connections that are not exclusive to the IMT.

Logical Implication Connections Relevant to the First Theorem of Logical Equivalence

In evoking connections relevant to the first theorem of logical equivalence, the interviewees tended to reference span, pivot positions, and linear combinations. Interestingly, several interviewees presented interpretations of span that were likely consistent with the formal definition of span, but interviewees rarely explicitly referenced the formal definition. That is, several interviewees were able to provide geometric interpretations of span or were able to describe span via linear combinations without explicitly saying the phrase “linear combinations.” For example, consider Will’s explanation of why two particular vectors span $\mathbb{R}^2$:

> **Will:** Because these two aren't scalars of each other, they're going in different directions. They each have their own x_1 and x_2 components. If they were the same, they'd just end up looking like that.

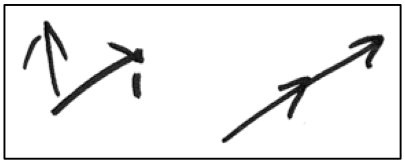


Figure **3**: Will provided a geometric description of what it means for two vectors to span $\mathbb{R}^2$. The illustration on the left represents an example Will provided of two vectors that span $\mathbb{R}^2$, while the illustration on the right represents an example Will provided of two vectors that do not span $\mathbb{R}^2$.

Seth provided an explanation that incorporated both matrix and geometric interpretations of span:

> **Seth:** If you had a matrix, let's take this one [Seth draws the 2×2 identity matrix], then this one would span all of $\mathbb{R}^2$ because no matter how you rearrange this, you can create – uh, I'll expand it [Seth then changes his matrix to the 3×3 identity]. So uh, this one can create,

> because you can multiply this by infinitely many scalars outside for each row, you can create infinitely many planes, like, if you think about this geometrically, planes in any coordinate system.

Jimmy appeared to allude to linear combinations, but also referenced a geometric interpretation of span:

> **Jimmy:** Well, for spanning, you want, uh. Every direction to be covered, every direction on the plane to be covered by some scale, er, some combination of those vectors, I think.

Jason explained that three particular vectors span $\mathbb{R}^3$ because "they go in different directions. They're not, uh, linear combinations of each other." While Jason referenced linear combinations, it was not in reference to the formal definition of span, but rather, as a description of what must be true of a set of vectors in order to span an entire space. Bill heavily alluded to linear combinations but did not explicitly reference linear combinations:

> **Bill:** Span is having the ability to make any vector within a space. You can, like I said, manipulate any piece of the outcoming vector. You can change it by changing one of the more, one of the scalar multiples along there, not scalar multiple, scalar weights along the way you go. In this case we did at x_1, x_2, x_3. If you could change each of those to then manipulate one of the vectors in the overall value within the system, you could then change the outcome. That goes into the span. If you can do that then it does span $\mathbb{R}^3$, it does span $\mathbb{R}$ whatever. It has the ability to reach any vector, any point within that space.

Bill's description of scalar weights and manipulating vector may provide evidence that he is describing linear combinations, although he does not explicitly reference linear combinations. Thus, Bill's interpretation of span is likely consistent with the formal definition of span, even if he cannot provide the formal definition.

It should be noted that while several students provided geometric descriptions of span, geometric interpretations were not heavily emphasized in class. They were briefly referenced from time to time, but concepts were never defined from a geometric perspective. Further, prior to span being defined, the class had discussed the problem of determining whether any vector in an $\mathbb{R}^n$ space can be expressed as a linear combination of a particular set of vectors. However, when span was formally defined, it was defined more generally as the set of all linear combinations of a set of vectors. Despite this, several interviewees appeared capable of determining whether a particular set of vectors spans an $\mathbb{R}^n$ space by determining whether the vectors were linearly independent, linear combinations of each other, or go in different directions. Thus, it is likely that students developed these alternative, yet mathematically correct, interpretations of span as a result of the inquiry-oriented activity previously described.

Logical Implication Connections Relevant to the Second Theorem of Logical Equivalence

Many of the connections the interviewees evoked relevant to the second theorem of logical equivalence involved pivot positions, linear independence, and basic and free variables. The interviewees tended to refer to basic and free variables in their logical implications more than any other concept; this was particularly interesting, as the interviewees from the previous semester tended to refer to pivot positions more than any other concept. The interviewees also appeared to have serious misunderstandings of the homogeneous equation. For example, Fred appeared to believe that *any* homogeneous equation can only have the trivial solution:

> **Interviewer:** If I had given you zeroes here instead of 3 and 2, would that still have a solution? That homogeneous linear system?

Fred: Yes, because homogeneous equation always have at least one solution, which is the trivial solution.

Interviewer: And what was the trivial solution again? Can you remind me one more time, what was that?

Fred: Trivial solution is $A\mathbf{x} = \mathbf{0}$, so zero is always the solution, for example $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Jason appeared to hold a similar view:

Interviewer: Can you define homogeneous equation for me? What does that mean?

Jason: It means there's only one solution. I can't remember what it was.

Seth appeared to confound the trivial solution with the homogeneous equation:

Interviewer: Do you remember what the trivial solution is?

Seth: Uh, it's when $A\mathbf{x} = \mathbf{0}$.

Cecily, who was incredibly close to relational understanding of the connection between pivot positions and linear independence, made a similar mistake:

Interviewer: Why is it that not having a pivot position in every column tells you that these columns cannot be linearly independent?

Cecily: Because if there's not a pivot position in every column, then it can have infinitely many solutions. And for it to be linearly independent, it can only have the trivial solution.

Interviewer: Okay. So what can only have the trivial solution?

Cecily: The matrix, the linear system.

Interviewer: Okay. So what is the trivial solution?

Cecily: That's $A\mathbf{x} = \mathbf{0}$, right?

I reminded Cecily that she was describing the homogeneous equation before asking her what the trivial solution is; she claimed she did not know. As these interviewees had misconceptions about the homogeneous equation, it is likely that any connections evoked that involved the homogeneous equation could only be instrumental; further, it suggests that these several interviewees have misunderstandings of the formal definition of linear independence. Indeed, similar to span, some students provided geometric descriptions of linear independence; Bill, for example, claimed that a particular set of vectors was linearly independent "because they can all point in different directions." Others essentially appeared to instead interpret linear independence through basic and free variables instead of the homogeneous equation. For example, consider Seth's explanation of linear independence:

Interviewer: How, how come if it has no free variables, that means it's linearly independent?

Seth: Well if it has no free variable, that means that there was a pivot in every column, which would mean that it would have no free variables, because there wouldn't be, like, say a 2 out here. And, uh. This vector would always have a solution.

Seth's response was not unique. Several other interviewees tended to refer to basic and free variables often in their descriptions of linear independence, as did many students on one of the class exams.

It should be noted that when we discussed the homogeneous equation in class, we did not do this through an inquiry-oriented activity; I believed that the concept did not warrant such an activity, as students in the pilot study and fall semester appeared to understand the homogeneous equation fairly well through a mixture of lecture and whole class discussion. Looking back at the day that we discussed the homogeneous equation in the spring semester, I noticed that we concluded our initial coverage of the homogeneous equation with the following discussion of a homogeneous matrix equation that only had the trivial solution:

Instructor: So, could I have free variables?

Student: No.

Instructor: Kay. I can't have any free variables. Why not? Why can't I have free variables?

Student: You'd have infinitely many solutions.

As this was how we concluded our initial coverage of the homogeneous equation, it is possible that some students essentially replaced the concepts of trivial and nontrivial solutions with basic and free variables. That is, students made an instrumental connection between the homogeneous equation and basic and free variables, and as they did not quite understand what the homogeneous equation is and when it has nontrivial solutions, they instead considered when the homogeneous equation would have free variables. Then, when the formal definition of linear independence was provided in terms of the homogeneous equation, they tended to view linear independence in terms of basic and free variables instead of the homogeneous equation. Thus, many students in this semester relied on their instrumental connection between the homogeneous equation and free variables in order to compensate for their lack of understanding of the connection between the homogeneous equation and linear independence, thus interpreting linear independence largely through basic and free variables. This was likely exacerbated by the aforementioned linear independence activity, in which students could refer to the familiar concept of basic and free variables to determine whether the sets were linearly independent or not. Students who had not come to rely as heavily on free variables likely developed more geometric interpretations of linear independence as a result of the linear independence activity, as the sets in the activity were in $\mathbb{R}^2$ and $\mathbb{R}^3$, which can be easily visualized. Once these students had developed a more geometric interpretation of linear independence, they may have felt that the formal definition was no longer necessary for an understanding of linear independence.

Conclusions

In light of the results from the interviews, it appears as though the inquiry-oriented activities that focused on span and linear independence were successful in creating opportunities for students to develop their own interpretations of span and linear independence. The role of geometric descriptions in the class was limited, yet several students developed interpretations of span that appeared to be more geometric in nature; further, these interpretations often heavily alluded to linear combinations while not explicitly referencing linear combinations in an algebraic sense. Regarding linear independence, the activity allowed students to reinforce interpretations of linear independence that heavily relied on basic and free variables; it also allowed students to develop geometric interpretations of linear independence that relied on the notion that linearly independent vectors are not linear combinations of each other.

While the inquiry-oriented activities were successful in creating opportunities for students to form their own interpretations of span and linear independence and how they relate to other concepts, the implementation of these activities could be improved. Students appeared to replace their understanding of the formal definition of span and linear independence with the understanding they developed as a result of the activities. This may have limited the students' ability to form logical implication connections involving these concepts and their formal definition. In retrospect, the instructor should have devoted time to exploring how these student developments relate to the formal definitions of span and liner independence. Investigating how the inquiry-oriented activities could be improved in this regard remains an avenue for future research.

References

Dorier, J.-L., & Sierpinska, A. (2001). Research into the teaching and learning of linear algebra. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study*. Netherlands: Kluwer Academic Publishers.

Harel, G. (1997). The Linear Algebra Curriculum Study Group Recommendations: Moving Beyond Concept Definition. In D. Carlson, C. R. Johnson, D. C. Lay, A. D. Porter, A. Watkins, & W. Watkins (Eds.), *Resources for Teaching Linear Algebra* (pp. 107–126).

Lay, D. C. (2011). *Linear algebra and its applications* (Fourth). Reading, Mass.: Addison-Wesley.

Rasmussen, C., & Kwon, O. N. (2007). An inquiry-oriented approach to undergraduate mathematics. *Journal of Mathematical Behavior*, *26*, 189–194.

Richards, J. (1991). Mathematical discussions. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 13–51). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Skemp, R. R. (1987). *The psychology of learning mathematics* (Expanded American edition). Hillsdale, N.J.: Lawrence Erlbaum Associates, Inc.