Leveraging Real Analysis to Foster Pedagogical Practices

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Although it is frequently a required course, many secondary teachers view real analysis as unnecessary and unrelated to teaching secondary mathematics. In accord with a proposed model for improving the teaching of advanced mathematics courses for teachers, we implemented a course that framed real analysis content by 'building up from' and 'stepping down to' teaching practice. In this paper, we describe how this model was implemented in a single module and analyze secondary mathematics teachers' engagement in and reflections on the desired pedagogical aims, which provide evidence that they saw what they learned in the real analysis module as being useful for informing their pedagogical practice.

Key words: Real Analysis, Secondary Teaching, Teacher Education, Pedagogical Practice

In the United States, prospective secondary mathematics teachers usually are required to take a substantial number of courses in advanced mathematics, including real analysis. However, while mathematical organizations believe that a mastery of advanced mathematics is important for teaching secondary mathematics (e.g., CBMS, 2012), research has also shown that completing courses beyond a fifth course in university studies – which is where advanced mathematics courses fall – yields very minimal gains in a secondary mathematics teachers' efficacy (Monk, 1994). Indeed, many students do not perceive any relevance between advanced mathematics and the teaching of high school mathematics (e.g., Zazkis & Leikin, 2010). In this paper, we address the following broad question: Given that prospective teachers are required to complete courses in advanced mathematics, how can we design these courses so that they productively inform teachers' future pedagogy? We look specifically at one module in a real analysis course designed according to this aim, and consider how the prospective and practicing teachers (PPTs) engaged in and reflected on a particular pedagogical practice.

Literature and Theoretical Perspective

A Model for Teaching Secondary Teachers Advanced Mathematics

From our point of view, the belief that completing a course in real analysis will improve a PPT's ability to teach secondary mathematics has been based on a traditional view of transfer from the cognitive psychology literature (e.g., Perkins & Salomon, 2002). More specifically, there is an assumption that as a byproduct of learning advanced mathematical content, PPTs will better understand secondary mathematics content and will consequently respond differently to instructional situations in the future – a tenuously presumed "trickle down" effect (Figure 1a). Given the notorious difficulties in achieving this type of transfer, it is not surprising that PPT's experience in real analysis often does not improve their teaching. In Figure 1b, we propose an alternative instructional model (Wasserman et al., 2016). This model is based on two premises. The first is the knowledge that PPTs learn should be inherently practice-based and applicable to the actual activity of teaching (e.g., Ball, Thames, & Phelps, 2008). The second is that PPTs will be more likely to make connections between real analysis and pedagogical practice if what they learn is situated within the context of teaching (e.g., Ticknor, 2012). Our model is composed of two parts: building up from practice and stepping down to practice. To *build up from (teaching) practice,* the real analysis content is preceded by a practical school-teaching situation. The building-up portion provides a context that sets the stage for the study of real analysis content in ways that are both relevant to teachers' practices and particularly well-suited to being learned in real analysis. The second part, *stepping down to (teaching) practice*, then uses the mathematical ideas from real analysis as a means to reconsider the pedagogical situation that began the module, as well as other relevant pedagogical situations. Stepping down to practice explicitly clarifies the intended mathematical and pedagogical aims. In between building up from and stepping down to practice, the real analysis topics are covered in ways true to its advanced character with formal and rigorous treatment, but the tasks make explicit what connections and implications these have for both secondary mathematics and its teaching.

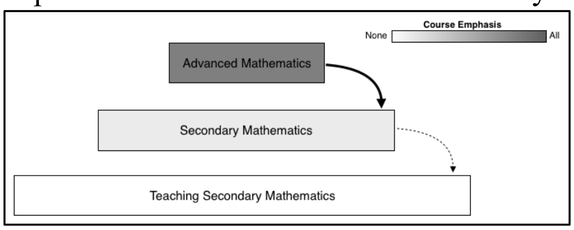

Figure 1a. Implicit model for advanced mathematics courses designed for teachers

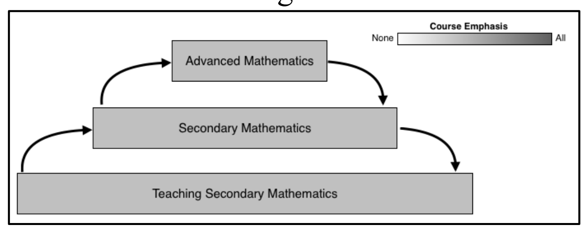

Figure 1b. Our model for advanced mathematics courses designed for teachers

An Example of our Model: Considering Derivative Proofs as "Attending to Scope"

In their professional work, teachers must explain content, practices, and strategies (e.g., TeachingWorks, 2016); we regard a particularly important facet of providing an explanation as being attentive to the scope of that explanation. For example, in trying to help elementary students understand subtraction, some teachers may state: "you cannot subtract a larger number from a smaller one." This explanation has a limited scope—it is only accurate when one is considering positive numbers – and can hinder students' future mathematical learning (e.g., Ball & Bass, 2000). Real analysis, with its rigorous proofs and its attention to the number sets to which a statement applies (e.g., the Intermediate Value Theorem requires the completeness axiom and consequently applies to **R** but not **Q**), is a domain in which attention to scope can be learned. In a real analysis course, students are expected to study and produce rigorous proofs of the common differentiation rules, such as the power rule and the product rule. (See, for instance, the textbooks of Abbott, 2015, and Fitzpatrick, 2006.) In the module that we describe, we use these real analysis proofs to highlight the importance of attending to the scope of a statement, and leverage them to help foster developing this desirable pedagogical practice.

Building up from practice. We began the module by presenting PPTs with the following situation: "Mr. Ryan teaches everything from Pre-Algebra to Calculus. The following scenes are snapshots from his classes at different times during the year." Using cartoons (which are not included for space purposes), two of Mr. Ryan's statements throughout the year were depicted: the *Exponents statement,* "Exponents are just repeated multiplication," and the *Power rule statement,* "If you see a function with an exponent, to take the derivative, you bring down the exponent to the front and subtract one from the exponent" [on the board was written $f(x)=2x^3$]. From this cartoon, PPTs were first asked to "Evaluate the pedagogical quality of each one of these explanations." We note that these two statements are *limited in scope*: exponents can only be viewed as repeated multiplication if the exponent is a positive integer, and the power rule only works for power functions. Next, students were shown a typical proof of the power rule for differentiation, using the binomial expansion of $(x+h)^n$. They were asked to identify for which sets the proof would be valid (**N, Z**, **Q,** or **R**), with the point being that this proof, which was likely familiar and is often presented to calculus students, is only valid for natural numbers.

Real analysis. The real analysis portion of the course consisted of the presentation of proofs of the power rule, the product rule, the quotient rule, the chain rule, and the inverse function rule. A progression of proofs of the power rule (for $f(x)=x^n$, $f'(x)=nx^{n-1}$) was presented that differed depending on the scope (i.e., for *natural*-numbered exponents, for (non-zero) *integer*-numbered exponents, etc.), and the proofs of the product rule, quotient rule, chain rule, and inverse function rule were provided so that they could be used for the power rule proofs.

Stepping down to practice. After the real analysis was presented, PPTs were invited to revisit the classroom scenario presented earlier. They were also asked on their homework to evaluate the pedagogical quality of two other explanations that were limited in scope, which we called the *Perimeter statement* ("The perimeter is just the sum of all the side lengths") and the *Add zero statement* ("Remember, to multiply a number by ten, just add a 0 to the end."). Finally, PPTs were asked write a journal entry in which they reflected on: "What, if anything, did you find helpful for your teaching in this week's class? If there were helpful aspects, specify in what ways they might influence your teaching – if nothing was helpful, explain why."

Methodology

Research Context, Participants, and Data Collection

We designed an experimental real analysis course in which each session was designed using the model presented in Figure 1b and described above. We implemented this course with 32 PPTs, 31 of whom agreed to participate in our research study. In this paper we focus on the Attention to Scope module, which took place across two 100-minute sessions. We collected and analyzed three sources of data: (i) we audio- or video-recorded all students collaboratively working on the module's activities; (ii) we collected their homework responses to the *Perimeter statement* and the *Add zero statement*; and (iii) we collected their reflective journal assignments for what they learned from these modules.

Analysis

We coded each source of data in the following way. For (i), to analyze PPT's in-class activity, we had recordings from five tables (T1, T2, T3, T4, T5), each containing about six students. When the groups were analyzing Mr. Ryan's exponent and power rule explanations, we used an open coding scheme in the style of Strauss and Corbin (1990) to capture the aspects of the classroom scenario to which the PPTs attended. When asked to cite the limitations of the proof of the power rule using the binomial expansion, we recorded each group's answers and the justification for their answers. For (ii), when coding the homework responses, we determine if the PPTs mentioned a limitation in scope for the statement and whether this limitation in scope was mathematically accurate. For (iii), we used an open coding scheme to document the category of responses that were present in PPT's reflective journal entries.

Results

We organize the presentation of results from our analysis in terms of their support for three particular claims: Claim 1) When evaluating the pedagogical quality of a teacher's statement, PPTs increased the attention they gave to the mathematical scope and limitations of a statement; Claim 2) PPTs valued the idea of attending to the scope and language of an explanation for their teaching; and Claim 3) Our model contributed to the goals of the module, particularly via the use pedagogical discussions to motivate the real analysis and vice versa.

To document Claim 1, we argue that PPTs showed limited attention the scope of Mr. Ryan's explanations at the start of the module, but that PPTs showed increased attention to scope on their homework assignment. When considering Mr. Ryan's explanations at the start of the module, we note that all tables were engaged with the task, and all evaluated the pedagogical quality of the exponents statement and the power rule statement negatively and gave reasons for their justification. At each table, a number of pedagogical concerns were raised. These included, for example, concerns about what students might understand 'repeated multiplication' to mean (e.g., "2x3x5x7" (T1)), and whether 'bringing down the exponent to the front' might be unclear (e.g., "It could be 23, not 6" (T3)). However, for some tables, this was the extent of their evaluations – focusing on the '*explanation* of the mathematics' and not the '*mathematical* aspects of the explanation.' Indeed, specifically in consideration of scope, only two tables (T3 and T4) attended to the limitations of scope for both statements, one table (T5) considered the limitation of scope with the exponents statement but not the power rule statement, and the other two tables (T1 and T2) did not attend to scope at all. That is, only 2 of 5 tables were consistent in their attending to the limited scope of both explanations. In contrast, however, on the homework assignment, both for the Perimeter and the Add Zero statements, all 31 PPTs correctly noted the limited scope of these statements. In addition, the quality of their attention to mathematical scope in the HW responses also increased. Even for the two tables that did so initially (T3 and T4), the exploration of the mathematical limitations of the statements was relatively narrow – they did not attempt to exhaust the possible scenarios. Each table only identified *one* instance where the power rule statement was limited (i.e., $\sin^2(x)$) – neither table discussed, for example, the derivative of e^x. In the HW exercises, however, the quality of the PPTs exploration of mathematical limitations was richer and more exhaustive. Both within individual responses as well as collectively across all PPTs, there was greater variety of limitations referenced (i.e., straight/curved, closed/open, exterior/interior lines, single/composite figures, 2D/3D) – indeed some of their discussions went beyond what was initially anticipated. We regard both the increased number of PPTs as well as the improved quality of responses as supporting Claim 1.

We document Claim 2 based on PPT's written reflections on what they learned from this module. In general, a common theme from PPT's written reflections was that they specifically valued the desirable pedagogical idea of attending to the scope and language of an explanation for their teaching – primarily addressing the mathematical precision of their language with students. Of the 27 PPTs who submitted reflections, 25 identified this idea as both: i) specifically stemming from the real analysis module; and ii) valuable for their teaching. We see both of these aspects in the following response that is representative of their reflections: "This lesson made me realize that as a teacher I must pay close attention to what I am saying. When I make statements that have errors, I need to know what loopholes or misconceptions held in my statement and be conscious of these as I create examples and answer questions" (S1). Here, it is worth reminding the reader that the PPTs were not obliged to say that they learned anything useful from the real analysis class (an option that some PPTs chose when reflecting on other modules). Within their statements, a few subthemes about implementation considerations arose: 1) scaffold definitions, beginning simple but getting increasingly rigorous (6 responses); 2) make sure explanations were not just procedural (5 responses); and 3) take into account the teaching context when considering the rigor and accessibility of explanations used (7 responses). In summary, we take this as evidence that PPTs saw pedagogical value in this module.

Lastly, we consider Claim 3 about the contribution of the model toward these aims. Notably, the data supporting this claim is anecdotal; however, we regard reflection on the model as important, and the responses from some students as suggestive about its contribution. First, we explore the possible value that the pedagogical situation may have added to the real analysis. Within PPTs' evaluation of Mr. Ryan's statements initially, pedagogical quality hinged somewhat on the mathematical scope and limitations. Thus, when transitioning to the real analysis proofs, PPTs appear to have given additional gravitas to considering mathematical limitations because of the related professional value, and their sense of the proofs may have been similarly tied to these limitations. As one instance, upon realization that the first power rule proof was limited to **N**, one student concluded: "This proof takes anything that I've ever believed in... Like here's a proof. Not anymore! Like this is the proof that I've always taught. And now, I'm like, everything about this is wrong" (T3). That is, the negative pedagogical evaluations appear to have prompted further mathematical motivation. Second, we explore the potential value that the real analysis may have added to accomplishing the desired pedagogical aims. Notably, the sequence of real analysis content explicitly modeled this attention to scope. And although one might do this without real analysis, at least some PPTs made this link and reflected on the value: "Seeing the connection between the analysis content and two very different concepts taught in high school was particularly useful.... the progression we took in the proofs from each set of numbers was a very elegant way of showing the different methods of proof, showing the flaws within each…" (S6). We see these – and other – comments as supporting the idea that the interaction between pedagogical discussion and real analysis was mutually beneficial to developing both.

Discussion and Conclusion

The analysis and reporting in this paper of one module from an experimental real analysis course – a single case study – sought to explore the broad issue of how advanced mathematics courses can be designed to inform PPTs pedagogical practice. In particular, the data from the study support the claim that, after engaging in the 'Attending to Scope' module from the real analysis course, the PPTs both increased their attention to (Claim 1) and valued (Claim 2) the desirable pedagogical practice of attending to the mathematical scope and limitations in teachers' explanations. By design, the real analysis content was both tightly connected to and framed by this pedagogical practice; however, as was evident from some of the tables of PPTs (i.e., T3, T4), one does not have to learn real analysis to be able to attend to the scope and limitations of secondary mathematics explanations. However, since a real analysis course already sort of inherently models this idea in both the precision of statements and progression of proofs, it seems sensible to exploit this connection for teachers. Indeed, the teachers in this study, overall, increased their attention to and valued this pedagogical practice. The anecdotal evidence for Claim 3 also seems to support at least one of the ways in which the model may have facilitated these goals. Thus, we see this as evidence that by framing real analysis content with pedagogical situations, in addition to learning real analysis, PPTs can also learn important teaching ideas. Further work studying how best to mathematically prepare secondary teachers is needed, including the degree to which this particular model is productive and/or needs refinement, and could help guide improved design and implementation of advanced mathematics courses for secondary teachers.

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