Developing Students’ Reasoning about the Derivative of Complex-Valued Functions with the Aid of Geometer’s Sketchpad (GSP)

Jonathan Troup
University of Oklahoma

In this paper, I share results of a case study describing the development of two undergraduate students’ geometric reasoning about the derivative of a complex-valued function with the aid of Geometer’s Sketchpad (GSP). My participants initially had difficulty reasoning about the derivative as a rotation and dilation. Without the aid of GSP, they could describe the rotation and dilation aspect of the derivative for linear complex-valued functions, but were unable to generalize this to non-linear complex-valued functions. Participants’ use of GSP, speech, and gesture assisted with discovering function behavior, generalizing how the derivative describes the rotation and dilation of an image with respect to its pre-image for non-linear complex-valued functions, and recognizing that the derivative is a local property.

Keywords: Amplitwist, Complex-valued function, Derivative, Dynamic Geometric Environments (DGEs), Gesture

Introduction

In the calculus reform era, a main goal was to develop students’ conceptual understanding of calculus by integrating algebraic and geometric reasoning (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Lauten, Graham, & Ferrini-Mundy, 1994; Meel, 1998). While similar research has been conducted in other mathematical content domains, there is less research in complex analysis, which allows potential for geometric reasoning. The purpose of this research was thus to address two research questions: What is the nature of students’ reasoning about the derivative of complex-valued functions, and what role does GSP play in developing students’ geometric reasoning about the derivative of complex-valued functions (i.e., the amplitwist)? For my work, I adopted the National Council of Teachers of Mathematics’ (NCTM, 2009) reasoning definition, which is “the process of drawing conclusions on the basis of evidence or stated assumptions” (p. 4), and is categorized as algebraic or geometric. Algebraic reasoning involves symbolic manipulation, and geometric reasoning involves spatial elements. I also refer to inscriptions, which Roth and McGinn (1998) define as “signs that are materially embodied … and because of their material embodiment, inscriptions (in contrast to mental representations) are publicly and directly available, so that they are primarily social objects” (p. 37).

Literature Review

In this section, I summarize literature on students’ reasoning in the realm of complex analysis, and the benefits of DGEs on students’ reasoning. As these domains draw on gesture as evidence for reasoning, I conclude this section by synthesizing related gesture research.

The Teaching and Learning of Complex Numbers

There has been a recent increase in educational studies focusing on complex numbers (Danenhower, 2006; Harel, 2013; Karakok, Soto-Johnson, & Anderson-Dyben, 2014;
Nemirovsky, Rasmussen, Sweeney, & Wawro, 2012; Panaoura, Elia, Gagatsis, & Giatilis; 2006; Soto-Johnson, 2014; Soto-Johnson & Troup, 2014. The results of these studies show that participants struggle to reason geometrically about complex numbers. In brief, Panaoura et al. found that high school students had difficulty transitioning between algebraic and geometric reasoning, while Danenhower showed undergraduate mathematics majors could not convert $a+ib \over c+id$ to exponential or Cartesian form. Applying Sfard’s (1991) duality principle, Karakok et al. saw that teachers displayed a process/object duality of the Cartesian form, but only an operational level of the exponential form. Harel (2013) noted that teachers were unable to attach a geometric meaning to the addition and multiplication complex numbers. However, reasoning via embodied cognition may help students reason geometrically about complex analysis. Nemirovsky et al. (2012) observed that through the usage of a “floor tile” representation of the complex plane and stick-on dots and string to represent complex numbers, pre-service teachers discovered multiplying by $i$ corresponds to a rigid 90° rotation. Soto-Johnson and Troup (2014) also found that undergraduates integrated their algebraic and geometric reasoning while drawing a representative diagram.

**Dynamic Geometric Environments**

In contrast with complex analysis research, research on educational technology is abundant. Some suggest that technology can cause harm by promoting overgeneralizations (Clements & Battista, 1992; Olive, 2000) or an over-reliance on technology (Salomon, 1990). Alternatively, technology can help students and teachers refine mathematical ideas (Arcavi & Hadas, 2000; Barrera-Mora & Reyes-Rodríguez, 2013; Heid & Blume, 2008; Hollebrands, 2007; Jones, 2000; Olive, 2000; Tabaghi & Sinclair, 2013; Vitale, Swart, & Black, 2014) without relying on mathematical authorities such as teachers or textbooks. DGEs such as Geogebra have been shown to help develop concepts related to real-valued differentiation (Hohenwarter, Hohenwarter, Kreis, and Lavicza, 2008; Ndlovu, Wessels, and De Villiers, 2010). Salomon (1990) suggested that DGEs may help students by providing multiple representations of mathematical objects. Hollebrands (2007) and Olive (2000) contend that GSP makes abstract ideas appear more concrete, which could help students ground reasoning in the physical environment. For example, Tabaghi and Sinclair (2013) found that while interacting with an eigenvector sketch in Sketchpad, students reasoned via gesture.

**Gesture**

Many (Alibali & Nathan, 2012; Goldin-Meadow, 2003; Keene, Rasmussen, & Stephan, 2012; Roth, 2001) believe “gestures can be used as a window into what students in a classroom are thinking” (Keene et. al., p.367). Others suggest diagrams and gestures inform each other (Châtelet, 2000; Chen and Herbst, 2013). For example, a vector could express the result of the multiplication of two complex numbers and a flick of the wrist could convey the associated rotation and dilation (Soto-Johnson & Troup, 2014). Some also state gesture and speech form a single, integrated system to support both visual and verbal content (Alibali & Nathan, 2012; Goldin-Meadow, 2003; Keene et al., 2012; Roth, 2001). For example, Goldin-Meadows found gesture aids memory and the ability to describe it, while restricting gesture hampers this ability. Some literature suggests gesture can transform over time from primarily representational to primarily pointing as students become more familiar with mathematical procedures (Alibali & DiRusso, 1999; Soto-Johnson & Troup, 2014; Garcia & Engelke, 2012; Marrongelle, 2007; Soto-Johnson and Troup, 2014). Garcia and Engelke (2012) also noticed that their undergraduate
participants gestured more frequently when they were stuck on a task. Vitale et al.’s (2014) participants initially gestured to remind themselves of a geometric concept, but transitioned into using gesture as a validation tool. The students additionally interacted with virtual representations of spatial gestures, which resembled their “real” gestures. Given that gesture may result from explorations with technology (Tabaghi & Sinclair, 2013) and that gesture can arise organically, it was critical to adopt a framework that would be sensitive to this phenomenon. Thus, I employed embodied cognition as my theoretical perspective, which allowed me to interpret my participants’ reasoning via gesture and actions taken within a DGE. In the next section I summarize my interpretation of embodied cognition.

Theoretical Perspective

Embodied cognition is mainly concerned with the relationship between reasoning and actions within the physical environment (Anderson, 2003), though the nature of this relationship differs between researchers. While some suggest embodied cognition is a tool for mental cognition, (Alibali & Nathan, 2012; Lakoff & Nuñez, 2000; Wilson, 2002), others dispense with mental models entirely by equating reasoning with body-based activity, such as is found in Nemirovsky’s (2012) fluid “realm of possibilities.” Under the second view, bodily experience is itself an inextricable part of the learning process. So, one view is concerned with mental models influenced by embodied action, while the other focuses on personal experience. These views do not seem entirely incompatible; an impersonal description of pain as “the firing of C-fibers” (Nemirovsky et al., 2012, p. 2) neither invalidates nor takes precedence over someone’s personal experience with pain. Similarly, postulations about the cognitive way the mind responds to bodily actions may not be contradictory to inferences made about a learner’s experience. I align more closely with this second viewpoint. Similar to Soto-Johnson and Troup (2014c), I interpret embodied cognition to include bodily actions taken within the physical environment. I believe perceptuo-motor activity (including DGE activities) can influence reasoning and that reasoning can influence bodily actions. Rather than postulate any cognitive mechanisms, I simply suggest a relationship between how my participants utilized DGEs, gestures, and inscriptions. Particularly, participants reasoned with GSP via construction and manipulation of geometric inscriptions and via the production of virtual gesture through mouse movement. Thus, I take this reasoning and participants’ usage of GSP, gestures, and inscriptions to be inseparable.

Methods

This report is a summary of my Ph.D. thesis. Thus, in this section, I describe some of the aspects of research performed, including participant selection, task design, and data analysis.

Setting and Participants

To help with participant selection and triangulation, I attended complex analysis class sessions related to the derivative. I video-recorded these classes and took notes paying particular attention to gestures employed. Four students agreed to participate in my study, who I refer to as Christine, Zane, Edward, and Melody. I conducted two sets of interviews, one with Christine and Zane, and another with Melody and Edward. Each group worked on tasks over four non-consecutive days. Each interview lasted two hours.
**Concept Analysis of Amplitwist**

Within the context of real-valued functions, the derivative function has a well-recognized geometric interpretation as the slope of a tangent line. However, as the graph of a complex-valued function is four-dimensional, the generalization of this concept is not straightforward. To help overcome this problem, one can represent the graph of a complex-valued function as two sets of axes: one for domain and one for range. Thus, graphing a complex-valued function can be represented as a transformation $f: C \rightarrow C$. In describing the derivative of a complex-valued function geometrically, Needham (1997) considers how a complex-valued function maps an extremely small circle centered about the point $z$ in the domain. “The length of $f'(z)$ must be the magnification factor, and the argument of $f'(z)$ must be the angle of rotation,” (p. 196) a concept that Needham refers to as an amplitwist. Needham further elaborates that the derivative of a complex-valued function provides a linearization that locally approximates the function, as “‘expand and rotate’ is precisely what multiplication by a complex number means” (p. 196).

**The Tasks**

The overall goal of the tasks I developed for my interview sequences was to encourage reasoning about the derivative of a complex-valued function as described by Needham’s (1997) concept of an amplitwist. I utilized tasks developed from a previously conducted pilot study. In Tasks 1 and 2, students followed instructions on a lab worksheet to construct the function $f(z) = z^2$ and $f(z) = e^z$ with the aid of GSP and predict how the function maps points, lines, circles, and the complex plane (see Appendix B). The goals of this task were for participants to establish proficiency with GSP and determine the mapping of circles under a complex-valued function. For Task 3 I prepared the linear complex-valued function $f(z) = (3 + 2i)z$ with a complex-valued derivative in order to reduce the likelihood that students reasoned that the real part of the derivative is a dilation factor and the imaginary part of the derivative is a rotation factor. I asked participants to describe their geometric reasoning about the derivative of a complex-valued function both with and without GSP, and to use $f(z) = (3 + 2i)z$ to demonstrate. I re-introduced GSP later to allow them to test their conjectures and continue to explore their reasoning. The purpose of this task was to help participants relate the magnitude and argument of the derivative to the way the linear function dilates and rotates a circle. In Task 4, I asked participants to generalize their reasoning from Task 3 to the functions from Tasks 1 and 2 as well as the functions $f(z) = z^2$, $f(z) = e^z$, and $f(z) = \frac{1}{z}$. The goal of this exercise was to support students’ efforts to generalize the geometric reasoning they developed in Task 3 to the general case. Finally, in Task 5, I asked students to determine the value of a derivative at a particular point for the rational transformation $f(z) = \frac{(2z+1)}{(z+i)(1-z)}$. I gave them only geometric information about a rational transformation, and asked them to use this information to reconstruct the algebraic formula. The goal of this task was to encourage students to develop reasoning about the derivative as a local property and to develop reasoning about points of non-differentiability as they relate to the amplitwist concept.

**Data Collection and Analysis**

To obtain data, I video-recorded class sessions the instructor deemed relevant. I recorded all interviews with both a camera to capture gesture and screen-capture software to record actions taken with GSP. My analysis began by watching the videos in conjunction with the screen-captured GSP recordings to determine where the participants appeared to be making progress.
toward a conception of the derivative of a complex-valued function as a local linearization. I transcribed all recorded gesture, speech, and usage of inscriptions, after which I coded lines as algebraic or geometric based on this data. Then, I wrote summaries of each day for each set of participants wherein I detailed the progression of selected events. Finally, I performed both a cross-case and within-case (Merriam, 2009; Patton, 2001) analysis on the written summaries, referencing both the Excel spreadsheet and the actual raw data as needed.

Results and Discussion

My findings suggest that GSP helped my participants generalize how the derivative describes the rotation and dilation of an image with respect to its pre-image for non-linear complex-valued functions. Furthermore, asking participants to construct an algebraic formula from geometric data served as a reminder that the derivative is a local property. For the first two tasks, I did not ask the participants anything about the derivative. Rather, I asked them to construct the function \( f(z) = z^2 \) for Task 1, and \( f(z) = e^z \). In the process, I answered their questions about how to use GSP, and asked them questions about how the function mapped various circles. During this questioning, both groups noted that circles mapped to other roughly circular shapes, and recalled that points “rotate” and “dilate” or “stretch” when multiplied by a complex-valued function. They did not appear to say anything about rotation and dilation in conjunction with the derivative of a complex-valued function, however. Rather, Christine states simply, “I don’t know like what slope means in complex world.”

When participants reasoned about the geometry of a linear complex-valued function \( f(z) = (3 + 2i)z \) in Task 3, they verbalized that the function maps a circle to another circle which is rotated by the argument of \( 3 + 2i \) and dilated by \( |3 + 2i| \) with respect to the original circle. While Melody and Edward made this observation in Task 3, Christine and Zane did not verbalize this same point until Task 4, although they did mention that the “stretch” and “rotation” factors do not change (see Error! Reference source not found. (a)). Melody and Edward observed, “the derivative is rotating this consistently wherever it is and then it’s expanding it out whatever the length of the derivative is. I guess we can see if that’s true” (see 1(b)).

Figure 1: \( f(z) = (3 + 2i)z \) transforms green input circle and spokes to blue output circle and spokes of corresponding color (a) and Edward transforms a circle under \( z \rightarrow 2z \) (b).

Later, during Task 4, Melody adds gesture to their description:

Melody: Like the center (points at (2,0)) would be at 2? So the center of the circle doesn't necessarily depend on the derivative. Like where the output one is located doesn't depend on the derivative (curls fingers slightly into claw (see Figure 2(a)), beats toward screen while moving arm counterclockwise in an upper circular arc) The output one is just the size (hand makes claw
shape, extends fingers outward and back in) of it and (twists hand first clockwise like a doorknob (see Figure 2(b)) then back counterclockwise) the rotation not the, like the location would depend on the z squared.

Figure 2: Melody produces a “claw-like” gesture for dilation (a) and produces a “doorknob” gesture for rotation (b).

One of the most interesting findings resulted from Task 5, wherein participants utilized geometric data to construct an algebraic formula for a rational function. In this task, Edward and Melody explicitly connected strange output behavior to non-differentiable points, stating, “that can’t be differentiable there...It’s weird” (see Figure 3). At this point, Edward was able to verbally reason geometrically about what made a point differentiable, by stating, “okay, so to know if this is differentiable, we want to kind of know when, where, goes to, small circle goes to small circle.” This utterance is an atypically precise geometric description of the fact that the derivative is a local property. It was during this task that participants most precisely characterized the derivative as a local property. Note that only Edward and Melody accomplished this task, as Zane and Christine progressed more slowly through the tasks.

Figure 3: Edward and Melody zero in on a non-differentiable point.

Overall, I answer the two research questions listed above in the following ways. To answer the first, I argue that my participants reasoned about the derivative of a complex-valued function via embodied cognition in three distinct ways. In particular, they grounded their algebraic and geometric inscriptions via gesture and speech as in Goldin-Meadow (2003) (see leftmost cycle in Figure 4), integrated their algebraic and geometric reasoning methods via these inscriptions as in Soto-Johnson and Troup (2014) (see center cycle in Figure 4), and grounded these reasoning methods in both the real and virtual environments as in Hollebrands (2007), Olive (2000), and Tabaghi and Sinclair (2013) (see rightmost cycle in 4. To answer the second question, I detail three developments in reasoning which arose during their work with GSP, and seemed critical to my participants’ reasoning about the derivative.
These three developments are as follows. First, my participants recognized a need for a geometric characterization of linear complex-valued functions, which was supported by grounding algebraic and geometric inscriptions via gesture and speech. Many of these inscriptions were displayed by GSP. This development allowed participants to begin extending their reasoning about real-valued functions to the complex-valued case. Second, my participants reasoned geometrically that linear complex-valued functions rotate and dilate every circle by the same amounts $\text{Arg}(f'(z))$ and $|f'(z)|$, respectively. Finally, the participants observed small circles map to small circle-like objects under any complex-valued function.

Concluding Remarks

Thus, it appears that in my case study, gesture, DGEs and speech really did work together to aid students in developing reasoning about the derivative of complex-valued functions. I conclude by describing some possible teaching implications and directions for future research.

Teaching Implications

The findings described above suggest a few implications for teaching the derivative of a complex-valued function. First, it demonstrates potential learning trajectory for students seeking to develop their geometric reasoning about the derivative of a complex-valued function. In particular, my students first developed reasoning about the geometry of lines in $\mathbb{C}$, then reasoned geometrically about a constant derivative in terms of rotation and dilation, and finally reasoned about the need to reason specifically about small circles. Second, it suggests that my students worked beneficially with DGEs when placed in pairs and allowed some opportunity for free exploration of related algebraic and geometric reasoning, as in Olive (2000). Finally, my research suggests that it may be helpful to direct students’ focus to key points over the course of their mathematical investigations, even when aided by a dynamic geometric environment (DGE).

Future Research

One possible direction for future research is to increase the breadth of these results by implementing dynamic geometric environments (DGEs) on a large scale in real classrooms and collecting quantitative data on student performance on tasks related to reasoning about the derivative of a complex-valued function as an amplitwist. Such research would theoretically allow these or related results to achieve some level of generality. Another possible direction is to increase depth of the results in this case study by iterating on the last task specifically. My participants reported beneficial effect from constructing algebraic information from exclusively geometric information from GSP, so further research on the effects of similar tasks for other students could prove highly interesting.
References


