Researchers have demonstrated the importance of covariational reasoning for students’ development of various mathematical ideas. Several researchers have also argued that creating and sustaining multiplicative objects is a necessary mental action to reason covariationally. In this report, we describe task-design principles we have found to be productive for investigating and supporting students’ construction of multiplicative objects and their covariational reasoning. Drawing on our research investigating students’ covariational reasoning, we include data that highlights how these principles have been productive in our research and teaching.

Key words: Task Design, Covariational Reasoning, Multiplicative Objects

Researchers have identified that quantitative and covariational reasoning are important to the development of mathematical ideas such as rate of change and function (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Confrey & Smith, 1995; P. W. Thompson, 1994), with students’ abilities to develop images of quantities’ magnitudes being critical to their reasoning covariationally (Carlson et al., 2002; Saldanha & Thompson, 1998). Specifically, these images afford students opportunities to construct multiplicative objects; this notion of covariation involves an individual sustaining an image of two quantities’ magnitudes such that the two quantities are coupled so that she “tracks either quantity’s value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value” (Saldanha & Thompson, 1998, p. 299). Researchers examining students’ covariational reasoning have designed tasks to provide students opportunities to reason covariationally—*City Travels* (Saldanha & Thompson, 1998), *Power Tower* (Moore, Silverman, Paoletti, & LaForest, 2014), *Over and Back* (A. G. Thompson & Thompson, 1996; P. W. Thompson, 1994), and adaptations of the *Bottle Problem* (Carlson et al., 2002; Johnson, 2015; Zeytun, Cetinkaya, & Erbas, 2010). From our ongoing work in developing models of students’ thinking and our interpretations of these researchers’ works, we describe task-design principles that we have found productive in investigating and supporting students’ construction of multiplicative objects and their covariational reasoning in small group and whole-class research and teaching settings. Our task-design principles are (a) avoid using quantities that are time or monotonic in the situation, (b) use shapes strategically to match or not match the graphical shape, (c) use different representational systems or orientations, and (d) use varying segment magnitudes to represent a quantity’s magnitude in a situation.

Background

In this paper, we draw on P. W. Thompson’s (2011) and Thompson, Hatfield, Yoon, Joshua, and Byerley’s (2016) extension of the definition of covariational reasoning offered by Saldanha and Thompson (1998). We note that covariation is not an inherent feature of a situation, graph, table, etc.; a student must conceive of a situation, graph, table, etc. as constituted by covarying quantities (P. W. Thompson, 2011). Thompson (2011) characterized covariational reasoning in
terms of an individual conceiving two quantities’ magnitudes (or values), $x$ and $y$, each varying (by intervals of size $\varepsilon$) over conceptual time, $t$. He used ordered pair notation, $(x(t), y(t))$, to denote the cognitive uniting of these two magnitudes, the result of which is a multiplicative object. Thompson’s use of multiplicative object stems from Piaget’s notion of “and” as a multiplicative operator (Inhelder & Piaget, 1958). The creation of such a multiplicative object is essential for conceiving displayed graphs as constituted by covarying quantities, and Thompson et al. (2016) explained that an individual’s conception of a multiplicative object is not restricted to or contained in a displayed graph. Instead, a graph is one way to represent a constructed multiplicative object. Thus, as we present our task-design principles, we focus on students’ construction of multiplicative objects primarily, but not exclusively, in the context of graphing.

With regard to previous recommendations on task-design principles, our tasks follow Gravemeijer and Doorman’s (1999) recommendation that tasks be experientially real—tasks involve a situation that students can imagine and understand in order to support conceptions of varying quantities. Additionally, our tasks include simplified versions of common situations, often in the form of dynamic videos or applets (Johnson, 2013, 2015; Saldanha & Thompson, 1998; P. W. Thompson, 1994). Both aforementioned principles are mentioned in Carlson, Larsen, and Lesh’s (2003) list of principles they used to structure covariational reasoning within a model-eliciting task. Lastly, although Johnson (2015) recommended sequencing tasks that support students in progressing from numerical to non-numerical reasoning, we avoid providing numbers in our tasks because quantitative reasoning is fundamentally non-numeric in nature (Smith III & Thompson, 2008). Moreover, we do not explicitly include non-numeracy in our task design principles because quantitative reasoning is principally non-numeric.

Task Background

Because of space limitations, we illustrate the task-design principles using two tasks: Going Around Gainesville (GAG) and Which One? (WO?). GAG is a modification of Saldanha and Thompson’s (1998) City Travels task. A student watches an animation of a car traveling around Gainesville on the way from Atlanta to Tampa (Figure 1). After we ask the student to describe the situation, we ask her to create a graph relating the car’s total distance traveled and distance from Gainesville during the trip (Part I). After the student addresses Part I, in which she chooses her axes orientation, we request a graph relating the car’s distance from Gainesville and distance from Atlanta on a given set of axes (Part II) (see Figure 1 for a normative solution to Part II).

![Figure 1: The Going Around Gainesville task and animation.](image)

The WO? task (Figure 2) is an adaptation of the Ferris Wheel task (Moore & Carlson, 2012). In this task, we present a student with an animation of a Ferris wheel rotating counterclockwise at a constant speed with a rider starting at the three o’clock position (see screenshot of animation in Figure 2a). We ask the student to describe the relationship between the height above the
horizontal diameter of the wheel and the arc length the rider has traversed. We then present the student with a simplified version of the Ferris wheel situation with the position of a single rider indicated by a dynamic point on a circle. Beside the situation, there are seven directed horizontal line segments. We inform the student that the topmost line segment (shown in blue) represents the arc length the rider has traveled counterclockwise from the initial three o’clock position. Students can change the length of this topmost segment by dragging point B or by clicking the ‘Vary’ button. As the position of the point on the circle (i.e., the rider) moves, the length of the topmost segment varies appropriately. We then ask the student to determine which of the six red segments (labeled in Figure 2b for the reader), if any, accurately represent the rider’s height above the horizontal diameter of the Ferris wheel as the rider’s arc length traveled varies. The design of these six segments is in Table 1; segment 1 is a normative solution to the task.

Table 1

<table>
<thead>
<tr>
<th>Seg.</th>
<th>(0 \leq B \leq \pi/2)</th>
<th>(\pi/2 \leq B \leq \pi)</th>
<th>(\pi \leq B \leq (3\pi)/2)</th>
<th>((3\pi)/2 \leq B \leq 2\pi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Increasing from 0 to 1 at a decreasing rate</td>
<td>Decreasing from 1 to 0 at a decreasing rate</td>
<td>Decreasing from 0 to (-1) at an increasing rate</td>
<td>Increasing from (-1) to 0 at an increasing rate</td>
</tr>
<tr>
<td>2</td>
<td>Increasing from 0 to 1 at a decreasing rate</td>
<td>Decreasing from 1 to 0 at a decreasing rate</td>
<td>Increasing from 0 to 1 at a decreasing rate</td>
<td>Decreasing from 1 to 0 at a decreasing rate</td>
</tr>
<tr>
<td>3</td>
<td>Decreasing from 1 to 0 at a decreasing rate</td>
<td>Decreasing from 0 to (-1) at a decreasing rate</td>
<td>Increasing from (-1) to 0 at an increasing rate</td>
<td>Increasing from 0 to 1 at a decreasing rate</td>
</tr>
<tr>
<td>4</td>
<td>Decreasing from 1 to 0 at a decreasing rate</td>
<td>Increasing from 0 to 1 at an increasing rate</td>
<td>Decreasing from 1 to 0 at a decreasing rate</td>
<td>Increasing from 0 to 1 at a decreasing rate</td>
</tr>
<tr>
<td>5</td>
<td>Increasing from 0 to 1 at a constant rate</td>
<td>Decreasing from 1 to 0 at a constant rate</td>
<td>Decreasing from 0 to (-1) at a constant rate</td>
<td>Increasing from (-1) to 0 at a constant rate</td>
</tr>
<tr>
<td>6</td>
<td>Increasing from 0 to 1 at a constant rate</td>
<td>Decreasing from 1 to 0 at a constant rate</td>
<td>Increasing from 0 to 1 at a constant rate</td>
<td>Decreasing from 1 to 0 at a constant rate</td>
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Task-Design Principles and Illustrations

In this section we provide a description of each of the design principles and examples of student activity highlighting the productivity of these principles. The students were pre-service secondary mathematics teachers at a large university in the southeastern U.S. and were enrolled in or had completed a content course in a secondary mathematics education program. All students had completed a calculus sequence and at least two additional mathematics courses (e.g., linear algebra, differential equations, etc.) with at least C letter grades. We collected data using semi-structured task-based clinical interviews (Clement, 2000; Goldin, 2000) and teaching experiments (Steffe & Thompson, 2000), a methodology in which task design is critical and rooted in developing models of student thinking and seeking to engender shifts in student
thinking based on those models; one’s hypothetical models of student thinking and shifts in student thinking drives her or his design of tasks. The purpose of this paper is to report on our task-design decisions, and thus we point the reader to our other works for more information about our methods in the context of particular studies (e.g., Moore, Paoletti, & Musgrave, 2013; Moore, Paoletti, Stevens, & Hobson, 2016).

Avoid using quantities that are time or monotonic in the situation

One of the key features of considering a graph as representing a multiplicative object is to understand a coordinate point simultaneously represents two magnitudes (oriented orthogonally in the normative Cartesian system); this construction is non-trivial (Thompson et al., 2016). Complicating the matter, researchers (Carlson, Larsen, & Lesh, 2003; Johnson, 2015; Leinhardt, Zaslavsky, & Stein, 1990) have argued that students often reason univariationally when creating graphs, particularly if time is one of the quantities under consideration (i.e., students consider changes in one quantity without explicitly coordinating a second quantity). In our work, we have identified that students have a propensity to reason about a quantity varying with respect to experiential time if the second quantity under consideration varies monotonically (Paoletti, 2015), especially if this quantity is represented on the Cartesian horizontal axis; due to the monotonic variation of the second quantity, the student need not maintain explicit attention to its variation. Hence, our first task-design principle is to prompt students to consider quantity pairs such that neither quantity is time or varies monotonically.

In GAG Part II, we ask the student to create a graph using distance from Gainesville on the horizontal axis and distance from Atlanta on the vertical axis; distance from Atlanta is equivalent to total distance traveled until the car reaches Tampa, at which point distance from Atlanta begins to decrease as the car travels back towards Atlanta. This part of the task asks students to represent two quantities that are not varying monotonically, which has enabled us to determine the extent to which students sustain an explicit coordination of the two quantities under consideration (i.e., the extent to which students maintain understanding their graph as a multiplicative object) or if they (tacitly) represent another quantity (i.e., total distance traveled or time) during their construction of the graph in Part II.

To illustrate, throughout her activity in GAG Part II, Alicia accurately described the relationship between distances from the two cities. However, as she graphed the relationship she accurately described, she drew a graph that monotonically increased with respect to both axes (Figure 3a); we infer Alicia tacitly imagined the quantity on the horizontal axis as increasing. With respect to the situation Alicia understood that neither quantity varies monotonically, but to represent graphically the relationship she perceived in the Cartesian coordinate system, Alicia needed to create and sustain a multiplicative object of both distances. Alicia eventually did conceive her graph as representing a multiplicative object, evidenced by when she pointed to the bottom-left segment in Figure 3a and stated, “our distance from Gainesville should be getting smaller instead of bigger”, and she was eventually able to adjust her graph (Figure 3b-3c) so that it represented the covariational relationship she understood to constitute the situation.

Figure 3: (a)-(c) Three stages of Alicia’s solution to GAG Part II.
**Use shapes strategically to match or not match the graphical shape**

Several researchers (Clement, 1989; Leinhardt et al., 1990; Monk, 1992) have reported on students incorporating conceived iconic elements (i.e., visual features) of a task situation into graphical representations. We highlight that, in that moment, a student is not constructing and sustaining a multiplicative object of the form we discuss. Rather, the student is forming figurative associations between a perceived situation and graph (P. W. Thompson, 2016). Hence, as a second task-design principle, we design tasks such that perceived shapes in the situation do match perceived shapes of normative graphs, do not match perceived shapes of normative graphs, and a combination of the two.

For instance, in *GAG*, the simplified road is composed of linear segments and a semicircle. While the car travels on the straight paths, the car’s distances from the two cities changes at a constant rate resulting in straight segments in the graph; while the car travels on the semicircular arc, however, the car’s distance from Atlanta is changing while the distance from Gainesville remains constant, resulting in another linear segment in the graph. Including these different components in the situation allows us to gain insights into whether a student is reasoning with quantitative relationships or engaging in iconic translations. Further, we incorporate the aforementioned combination to examine whether students generalize such associations (e.g., “shapes always match graph” or “shapes never match graph”).

To highlight the utility of this task-design principle, consider Alicia’s activity in *GAG* Part I. After accurately describing the relationship between the car’s distances from the two cities when drawing the segment from the vertical axis, Alicia engaged in an iconic translation as she drew a semicircular arc in her graph (Figure 4a). It was after creating the graph that Alicia thought about the result of that activity as a multiplicative object; she indicated that, to her, there was an increase and decrease in the distance from Gainesville while traveling along the semicircular path. Alicia then put concerted effort into justifying this part of her graph by comparing this claim with her image of the situation. Only when carefully attending to the car’s distance from Gainesville in the situation, measuring it with a ruler (Figure 4c), did she conceive that this quantity did not change on the semicircle and replaced the arc in her graph with a straight segment (Figure 4b). She stated, “They’re all the same! Why did I think it was changing? That’s the radius…. So, we should be the same distance from Gainesville”. Thus, Alicia compared the relationship she conceived her graph as representing to the relationship she constructed in her image of the situation, which resulted in her conceiving the same relationship as constituting both the graph and situation.

![Figure 4](image_url)

**Figure 4**: Two stages of Alicia’s solution to *GAG* Part I (a-b) and her measurement work (c).

**Use different representational systems or orientations**

Previous researchers (e.g., Moore et al., 2016; Moore et al., 2014; P. W. Thompson, 2016) have illustrated that students’ ways of thinking for graphs are often dominated by figurative activity (e.g., graphs “pass the vertical line test” or graphs are drawn left-to-right). This outcome
can be explained by students having infrequent opportunities to construct images of covarying quantities and their having repeated, and possibly exclusive, experiences with graphs that have those figurative attributes (Carlson et al., 2002; Moore et al., 2016; Smith III & Thompson, 2008). Thus, for our third task-design principle, we suggest having students use various orientations of coordinate systems (e.g., changing axes orientations) or entirely different representational systems (e.g., alternative coordinate systems and representational systems like the ones described in the next principle). This design principle has supported us in gaining insights into the extent that students’ actions are dominated by figurative activity or coordinating quantities. Moreover, we have been able to gain insights into the extent that students construct and sustain multiplicative objects across a variety of representational systems.

To illustrate, students may, from our perspective, be reasoning covariationally about two quantities in GAG Part I, but when given GAG Part II, they hesitate to draw a vertical line (see Figure 1) because they claim graphs cannot “look” like that. Other examples from GAG included students having difficulty constructing a relationship that is represented by a graph drawn “right-to-left”. We have shown such issues to be problematic for students as they create graphs, particularly if the students do not persistently focus on covarying quantities (Moore et al., 2016). Further, we have found that using a multitude of coordinate systems supports students in reasoning covariationally in order to conceive graphs in different coordinate systems as representing an invariant relationship between quantities (Moore, Paoletti, & Musgrave, 2013). Hence, like the other principles, using different coordinate systems or orientations supports us in differentiating students who are conceiving of graphs as representing a multiplicative object versus those whose thinking is dominated by figurative thought; when changing coordinate systems or orientations, figurative aspects of a function’s graph typically change.

**Use varying segment magnitudes to represent a quantity’s magnitude in a situation**

The three preceding task-design principles have been helpful in allowing us to have a sense of the students who are reasoning covariationally within and across situations and graphical representations. As previously mentioned, constructing a multiplicative object does not require a coordinate system (P. W. Thompson et al., 2016). By working with varying magnitudes instead of prompting a student to create a graph (e.g., WO?), we gain insights into students’ reasoning while minimizing the influence of the ways of thinking they have developed for graphs (e.g., iconic translations, issues of function/dependency, ways of thinking based in figurative thought). Specifically, we are able to gain insights into the extent that a student constructs and sustains a multiplicative object with respect to a situation and the displayed magnitudes. Moreover, we are able to gain related insights into students’ ways of reasoning about graphs by scaffolding tasks to support students in orienting the segments orthogonally, constructing a coordinate point, and imagining a trace representing all instantiated pairs of the covarying magnitudes (as seen in the “finger tool” explained in Lima, McClain, Castillo-Garsow, and P. W. Thompson (2009); P. W. Thompson (2002)).

Consider Lydia’s reasoning in the WO? task. When referencing the first quarter rotation of the Ferris wheel (Figure 2a), she explained, “[A]s the arc length is increasing… [the] vertical distance from the center is increasing… but the value that we’re increasing by is decreasing.” After providing this accurate description of the situation, Lydia moved to select the appropriate segment in the WO? task (Figure 2b). She eliminated all segments except 1 and 5 (Figure 2b). Focused on the top of the Ferris wheel, Lydia said, “I think it is [segment 5], because it is decreasing at the same rate that I am increasing [referring to her moving the blue segment].” Despite having potential constraints from a graphing task removed, Lydia reasoned about the
segments in a way that was not consistent (from the researchers’ perspective) with her reasoning about the quantities in the situation. Lydia’s contradictory statements in this task illustrate students’ difficulties with constructing and sustaining a covariational relationship across representational systems that do not necessitate also constructing a coordinate point.

As Lydia continued in her teaching experiment, having repeated opportunities to reason about magnitudes in non-graphical situations was beneficial in her coming to understand graphs as representations of multiplicative objects. For instance, on GAG Part II, Lydia could not initially construct a graph as requested. However, she then imagined horizontal segments representing the varying magnitudes of each of the distances. As the animation played, Lydia used her pen tips, both oriented horizontally, to indicate how each quantity’s magnitude varied (Figure 5a). Then, she engaged in the same activity but with her pen tips oriented orthogonally (Figure 5b); this activity supported her imagining and drawing a trace of the graph that she understood as uniting the two magnitudes into a point and trace (i.e., a multiplicative object) à la the “finger tool”).

(a) (b)

Figure 5: Lydia reasoning with (a) horizontal magnitudes and (b) orthogonal magnitudes on GAG Part II.

Discussion

We presented several task-design principles intended to afford students opportunities to reason covariationally as they construct, maintain, and represent multiplicative objects in various representational systems. We have found that tasks designed with these principles provides students repeated opportunities to construct and compare relationships between quantities across a multitude of representational activities. Moreover, these design principles afford teachers opportunities to gain insights into their students’ propensities to reason covariationally as well as to perturb students who engage in reasoning that is not attentive to covariational relationships. Hence, we believe that tasks designed using these principles will provide students with intellectual need (Harel, 2007); students will find covariational reasoning to be productive when engaging in these tasks. We conclude by emphasizing that we have found the last design principle most productive in our work with undergraduate students, particularly because of students’ propensity to reason about graphing in ways that do not entail quantitative or covariational reasoning.

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References


