Second Semester Calculus Students and the Contrapositive of the Nth Term Test

David Earls
University of New Hampshire

Little is known about the difficulties second semester calculus students have determining series convergence, and why students have such difficulty. This report seeks to add to the existing literature on series by analyzing second semester calculus student responses to a multiple choice item that involves the use of the contrapositive of the nth term test. We frame our discussion in terms of what these answer choices might say in terms of student concept images of series and sequences. We also analyze what prerequisite knowledge might help students be more successful in answering questions about series and sequences typically seen in a second semester calculus course.

Key words: Calculus, Nth Term Test, Series, Sequences, Contrapositive

Researchers have called for more research in the area of infinite series (González-Martín, Nardi, & Biza, 2011). Some of the existing literature on series includes the role infinity plays on students’ understanding of series (Sierpińska, 1987), student understanding of definitions (Roh, 2008; Martínez-Planell, R., Gonzalez, A., DiCristina, G., & Acevedo, V., 2012), student’s beliefs about their role as a learner and the relationship between these beliefs and approaches to solving convergence problems (Alcock & Simpson, 2004; Alcock & Simpson, 2005), how series are introduced to students (González-Martín, Nardi, & Biza, 2011), the difficulties students have accepting that comparison tests can be inconclusive (Nardi and Iannone, 2001), and the development of a framework used to analyze student errors determining the convergence of series (Earls & Demeke, 2016).

This preliminary report seeks to add to the existing literature on series convergence by analyzing second semester calculus student solutions to a problem that could be solved using the contrapositive of the nth term test. The report is part of a larger dissertation study that seeks to determine the difficulties students have finding the convergence of sequences and series in second semester calculus, and in what ways these difficulties are related to prerequisite knowledge students’ instructors might expect their students to have from precalculus and first semester calculus prior to entering the course.

Conceptual Framework

Tall and Vinner (1981) use the term “…concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (p. 152). A student’s concept image of a particular concept usually results from working with examples and non-examples (Vinner & Dreyfus, 1989). The concept image for concepts that do not have graphical components or have weak graphical components will include symbols, formulas, and associated properties (Vinner & Dreyfus, 1989).

Tall and Vinner (1981) define concept definition as “…a form of words used to specify that concept” (p. 152). They also differentiate between a student’s formal concept definition and a personal concept definition. A formal concept definition is the definition of a concept that is
agreed upon by the mathematical community. A personal concept definition, however, might be constructed by the student and could change over time.

Although distinct, concept images and concept definitions are intricately related. Tall and Vinner (1981) describe this relationship by saying, “for each individual a concept definition generates its own concept image” (p. 153). In other words, the words used to describe a particular concept generate a mental image associated with the concept. As an example of this relationship, consider the function concept. The formal concept definition of function can be described as a relation between two sets where each element of the first set is assigned exactly one element of the second set. However, a student studying functions might not remember this definition, and the concept image for the student might include the idea that a function must be given by a rule or formula.

In the discussion that follows, this paper describes what one multiple choice question might say about students’ concept images of sequences and series in general and the nth term test in particular.

**Research Methodology**

The targeted population for this study is second semester calculus students enrolled at a large research university in the northeastern United States. One hundred seventy-nine students responded to an anonymous six question multiple choice survey on sequences and series with a cover sheet. The cover sheet asked students to list their previous three mathematical courses, whether or not they had experience with sequences and series prior to entering the course, age, year, gender, race, and expected grade in course.

The main research aims of the full dissertation study are to (1) determine what misconceptions of sequences and series are revealed when students solve problems on sequences and series typically seen in a second semester calculus course, (2) determine what ways, if at all, these misconceptions relate to the prerequisite knowledge students are expected to have prior to starting a second semester calculus course, and (3) determine what additional knowledge or conceptualization of sequences and series students might need to be successful in second semester calculus courses.

This preliminary report focuses on one problem on the multiple choice test, a problem that focused on students’ understanding of the nth term test and its contrapositive. This question was chosen for this report because more students answered this problem incorrectly than any of the others:

**Instructions:** Please Circle The Letter of The Best Response.

Suppose that you know that

\[ \sum_{n=1}^{\infty} b_n = 3 \]

What can you say about \( \lim_{n \to \infty} b_n \)?

A. \( \lim_{n \to \infty} b_n = \infty \)
B. \( \lim_{n \to \infty} b_n = 0 \)
C. \( \lim_{n \to \infty} b_n = 3 \)
D. We can’t say anything about \( \lim_{n \to \infty} b_n \)
Recall that the nth term test states that, if \( \lim_{n \to \infty} a_n \neq 0 \), then \( \sum_{n=1}^\infty a_n \) diverges. Consequently, the contrapositive of this statement tells us that if the series converges, then the limit of the sequence must go to 0. Therefore, choice B is the correct answer.

This question was recommended by a mathematics professor with many years of experience as a second semester calculus instructor. Distractors were chosen based on responses given to this question during interviews in a pilot study. This choice was made based on Kehoe’s (1995) recommendation that any multiple choice assessment should have three to four well written answer choices. All of the questions on the multiple choice test were reviewed for appropriateness (as in, typical problems that a second semester calculus student should be able to solve) and correctness by a mathematics graduate student, a mathematics education graduate student, and an experienced mathematics professor.

Data was analyzed using first exploratory data analysis (EDA) followed by confirmatory data analysis (CDA). Exploratory analysis involves looking at variables individually, then two at a time, and then multiple variables at a time. Behrens (1997) recommends starting with EDA before moving to CDA because EDA allows researchers, “…to find patterns in the data that allow researchers to build rich mental models of the phenomenon being examined” (p. 154).

During EDA, some hypotheses developed. For example, it appeared as though students that had experience with sequences and series coming into the course performed better on this question than those that lacked experience. The purpose of CDA is to test this and other hypotheses using statistical significance tests such as a chi-squared test, or Fisher’s 2-tail test. These two statistical tests were chosen based on recommendations from the Institute for Digital Research and Education website (IDRE, n.d.). All statistical tests were performed using the JMP software.

**Preliminary Results**

Only 48 of the 179 students, or just over 26 percent, chose answer choice B, the correct answer to this question. Ninety-two students, which is over 50 percent, chose answer choice C, with seven students, under four percent, choosing answer choice A and 32, about 18 percent, choosing answer choice D.

Students that said they had experience with sequences and series in a previous course performed better on this question than those who said they had no experience. Sixty-eight students said they had experience with sequences and series prior to their second semester calculus course, and 111 said they had no experience. Unfortunately, the sample sizes were too small to determine if there was statistical significance within each answer choice. Consequently, I instead considered whether or not the students answered the question correctly.

Of the 68 students with prior experience, 25 students, or just over 36 percent, answered the question correctly. Of the 111 students with no prior experience, only 23, or just over 20 percent, answered the question correctly.

Fisher’s two-tail test revealed a p-value of .0238, indicating that this difference is statistically significant. Likelihood ratio and Pearson chi squared tests were also performed and revealed p-values of .0198 and .0187, further indicating the likelihood that the difference in performance between students with prior experience and those without prior experience is statistically significant.

The full contingency table from JMP is shown below to give a visual representation and more detail on what was presented above. The no/yes on the left side of the table indicates
whether or not students had experience with sequences and series prior to entering the course. The no/yes on the top of the table indicates whether or not students answered the question correctly:

**Figure 1: Contingency Table – Question six correct by experience**

<table>
<thead>
<tr>
<th>Experience?</th>
<th>Question 6 Correct?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>no</td>
</tr>
<tr>
<td>no</td>
<td></td>
</tr>
<tr>
<td>Count</td>
<td>88</td>
</tr>
<tr>
<td>Col %</td>
<td>49.16</td>
</tr>
<tr>
<td>Row %</td>
<td>67.18</td>
</tr>
<tr>
<td>yes</td>
<td></td>
</tr>
<tr>
<td>Count</td>
<td>43</td>
</tr>
<tr>
<td>Col %</td>
<td>24.02</td>
</tr>
<tr>
<td>Row %</td>
<td>32.82</td>
</tr>
<tr>
<td>Total</td>
<td>131</td>
</tr>
</tbody>
</table>

**Tests**

<table>
<thead>
<tr>
<th>Test</th>
<th>N</th>
<th>DF</th>
<th>-LogLike</th>
<th>RSquare (U)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>179</td>
<td>1</td>
<td>2.7146923</td>
<td>0.0261</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Test</th>
<th>ChiSquare</th>
<th>Prob&gt;ChiSq</th>
</tr>
</thead>
<tbody>
<tr>
<td>Likelihood Ratio</td>
<td>5.429</td>
<td>0.0198*</td>
</tr>
<tr>
<td>Pearson</td>
<td>5.531</td>
<td>0.0187*</td>
</tr>
</tbody>
</table>

**Fisher's Exact Test**

<table>
<thead>
<tr>
<th>Prob</th>
<th>Alternative Hypothesis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left</td>
<td>0.9939*  Prob(Question 6 Correct?=yes) is greater for Experience?=no than yes</td>
</tr>
<tr>
<td>Right</td>
<td>0.0153*  Prob(Question 6 Correct?=yes) is greater for Experience?=yes than no</td>
</tr>
<tr>
<td>2-Tail</td>
<td>0.0238*  Prob(Question 6 Correct?=yes) is different across Experience?</td>
</tr>
</tbody>
</table>

**Discussion**

There are several plausible explanations for answer choice C, which over 50 percent of the population thought was the correct answer. Note that the value for the sequence convergence in answer choice C is the same as the value of the series convergence. It is possible that students are having difficulty understanding the difference between sequence notation and series notation. In other words, their concept image of sequences and series may include the fact that the symbols are interchangeable. A review of existing literature on semiotics confirms that students have a difficult time reading and understanding symbols in mathematics (Marjoram, 1974; Chirume, 2012; Earle, 1977). However, more research is needed to understand the role of symbols on student concept images of sequences and series.

Another possible reason for answer choice C is that students don’t see the difference between sequences and series because of their use in everyday conversation. That is to say, in the English language, the words “sequence” and “series” are interchangeable. As an example, consider the sentence, “A long sequence of events has led me to a career in mathematics education.” Replacing the word ‘sequence’ with the word ‘series’ leaves the meaning of this
sentence unchanged. However, sequences and series are very different in mathematics. Perhaps further research can explore the effect of the English language on student concept images of sequences and series.

The results above indicate that students that had some prior experience with sequences and series performed significantly better than students whose first experience with sequences and series came in this particular second semester calculus course. This prior experience, however, varied greatly. Some students noted on the cover sheet that they had seen sequences and series in BC calculus in high school. Others said that they were familiar with finite series from high school precalculus courses. Still others noted that they recognized series from the definition of the integral in a first semester calculus course. Regardless of the level and depth of experience, some exposure to sequences and series prior to entering the course seemed to have helped. It is worth noting, however, that even among the students with prior experience, most students, about 64 percent, still answered the question incorrectly.

It is difficult to know exactly how the concept images of those who answered this question correctly differ from those that did not answer the question correctly. It does appear, however, that the concept images of those that answered the question correctly may include a proper understanding of the nth term test and its contrapositive. Those that did not answer the question correctly appear to have an incomplete concept image of the nth term test. In particular, it appears as though they do not recognize the contrapositive of the nth term test. The difficulties that students had in this problem with the contrapositive of the nth term test are consistent with existing literature describing student and teacher difficulties understanding the logical equivalence of a statement and its contrapositive (Gregg, 1997).

Questions

I plan to continue the statistical analysis of the other problems on the multiple choice assessment. As this is part of a larger dissertation study, I also intend to analyze the transcripts of students that solved this problem during interviews. To help with further analysis, I would like my presentation to receive feedback on the following questions:

1. What other correlations do you think I should look for as I analyze the other multiple choice items?
2. What other methods can I use to further analyze student concept images of the nth term test aside from the transcript analysis?
3. Are there any other methodological suggestions you have that might help enhance this study and future research projects on student concept images of sequences and series?

Implications

Moving forward, further research can be done to examine the effect of semiotics and word meanings in the English language on student concept images of sequences and series. Future research can also examine the understandings of sequences and series that students have in an undergraduate real analysis course, and this can be compared to the misconceptions of second semester calculus students. Results from studies such as the larger dissertation study have potential implications for the development of curriculum materials and teaching strategies that can be used to strengthen student concept images.
References


