A Student's Use of Definitions in the Derivation of the Taxicab Circle Equation

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Research shows that by observing properties and making conjectures in other geometries, students can better develop their understanding of concepts in Euclidean geometry. It is also known that definitions in mathematics are an integral part of understanding concepts, and are often not used correctly in proof or logic courses by students. APOS Theory is used as the framework in this preliminary data analysis to determine one students' understanding of certain definitions in Euclidean and Taxicab geometry, and her use of these definitions in deriving an equation for a circle in Taxicab geometry.

Key Words: Definitions, Geometry, Taxicab, Geometrical Reasoning

Introduction

It has been found that commonalities exist in higher-level math courses regarding students' inability to properly complete tasks involving definitions (Edwards & Ward, 2004), despite the expectations held for students enrolled in such courses. Edwards and Ward (2004) state that there were misconceptions in students' understanding of "the very nature of mathematical definitions, not just from the content of the definitions," (p. 411). In the context of geometry, since the properties of geometric figures are derived from definitions within an axiomatic system, it is important to note that a figure is "controlled by its definition," (Fischbein, 1993, p. 141).

In college geometry courses, Euclidean geometry and its axiomatic system is deeply studied, but other axiomatic systems receive little consideration (Byrkit, 1971; Hollebrands, Conner, & Smith, 2010), although research shows that by exploring concepts in non-Euclidean geometry, students can better understand Euclidean geometry (Dreiling, 2012; Hollebrands, Conner, & Smith, 2010; Jenkins, 1968). For example, Dreiling (2012) found that "through the exploration of these 'constructions' in taxicab geometry…[students] gained a deeper understanding of constructions in Euclidean Geometry." (Dreiling, 2012, p. 478).

For this preliminary report from the larger research study, we present results and discussion on the following research question: What is the learning trajectory one student followed to accommodate her understanding of *distance* and *circle* in Taxicab geometry?

Theoretical Framework

As a constructivist framework, APOS Theory is based on Jean Piaget's theory of reflective abstraction, or the process of constructing mental notions of mathematical knowledge and objects by an individual during cognitive development (Dubinksy, 2002). In APOS Theory, there are four different levels of cognitive development: Action, Process, Object, and Schema. In addition, there are mechanisms to move between these levels of cognitive development, such as interiorization and encapsulation. An Action in APOS Theory is when a student is able to transform objects by external stimuli, performing steps to complete this transformation. As a student reflects on an Action and has the ability to perform the Action in his or her head without external stimuli, we refer to that as an interiorized Action and call it a Process. Once a student is able to think of this Process as a whole, viewing it as a totality to which Actions or other

Processes could be applied, we say that an Object is constructed through the encapsulation of the Process. Finally, the entire collection of Actions, Processes, Objects, and other Schemas that are relevant to the original concept that form a coherent understanding is called a Schema (Dubinsky, 2002).

Methodology

This research study was conducted in a College Geometry course during Fall 2016, which has an introduction to proof course as a prerequisite. Since it is a cross listed course, there were seven undergraduate and 11 graduate students enrolled in the course, many of whom were pre-service teachers. The study is defined as a teaching experiment, as described by Cobb and Steffe (2010) and Steffe & Thompson (2000), which consisted of sessions of instruction, followed by individual interviews. The textbook used in the course was College Geometry Using the Geometer's Sketchpad (Reynolds & Fenton, 2011), which is written on the basis of APOS Theory. The material of the course covered concepts and theorems often seen in a College Geometry course, with Taxicab geometry included at the end of the semester. Videos from in-class group work and discussion, as well as written work from the semester were collected. After the semester, semi-structured interviews were conducted with the 15 of the 18 students enrolled in the course who volunteered. We focus our attention in this paper to one student and some of her answers to the interview questions, as they provided good insight as to how students transfer definitions to a new context. The following questions are relevant to this paper, and are a subset of the questions asked during the interview:

1. Define and draw an image (or images) that represents each of the following terms, however you see fit: Circle, Distance.
2. For any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$
   (i) Euclidean distance is given by $d_E(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
   (ii) Taxi distance is given by $d_T(P, Q) = |x_2 - x_1| + |y_2 - y_1|$
   Using the grids below, illustrate each of these two distances. Be as detailed as possible in labeling them.
3. Using the grids below, sketch the following circle in both geometries: Circle with center at $C(3,3)$ and radius $r = 2$.

These questions were used in the analysis for this preliminary report, since they help to identify student understanding of *distance* and *circle*, and the possible pathway a student takes to transfer and possibly modify her definitions to new situations. Specifically, for this paper, we focus on how a student used her definition of *distance* and *circle* in her attempt to derive the algebraic representation of a Taxicab circle.

A genetic decomposition is defined as a "description of how the concept may be constructed in an individual's mind," (Arnon et al., 2014, p. 17). A preliminary genetic decomposition was developed for this study to identify the development pathway students may follow to derive (or understand the derivation of) this equation (see Figure 1). To specify, *geometric representation* includes any sketch or drawing in addition to the students' verbal description of the definition, unless they specifically state the definition is an equation.

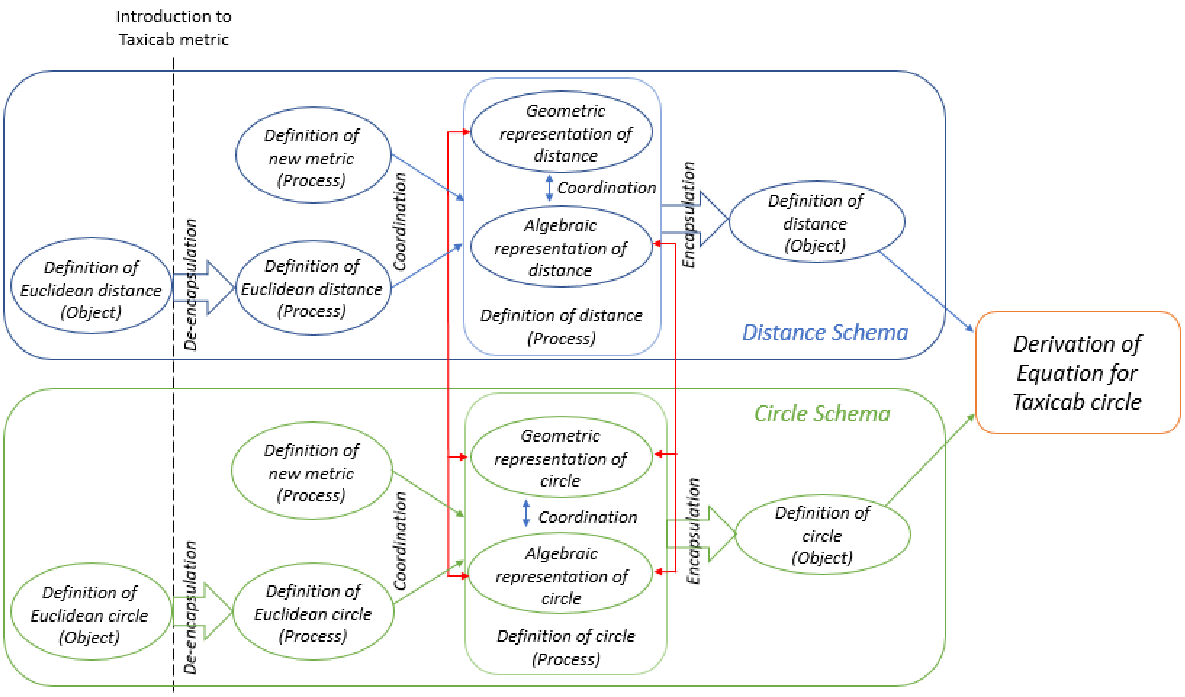

Figure 1: Preliminary genetic decomposition.

Based on our own understanding of historical development of these concepts, along with existent research results, we partitioned the concepts of *distance* and *circle* in terms of their schemas, and illustrated how students develop this equation in Taxicab geometry. As illustrated in Figure 1, we propose that in dealing with concepts of *distance* and *circle* in these two geometries for this task, students exhibit an interplay between two schemas. For the sake of length, we omit a full explanation of Figure 1.

Preliminary Results

We provide the APOS Theory based analysis of one student's answers, Nicole, as it corresponds to this preliminary genetic decomposition.

Geometric Representation of Distance

Nicole stated, "I know with Taxicab I can't just look at the distance as this from A to B, I have to go…up and around I mean up, like I'm using the road and not going through," and "there are certain steps I have to take or a certain route…" Illustrations of her conception of these metrics can be found in Figure 2. Later, in responding to a question about her Taxicab circle, Nicole was able to use a definition of Taxicab distance to justify that every point on her diamond shaped Taxicab circle was equidistant from the center. Therefore, she clearly exhibited an object conception of *geometric representation of distance* in Taxicab geometry since she was able to think of it as a totality and apply an action (comparison) to it.

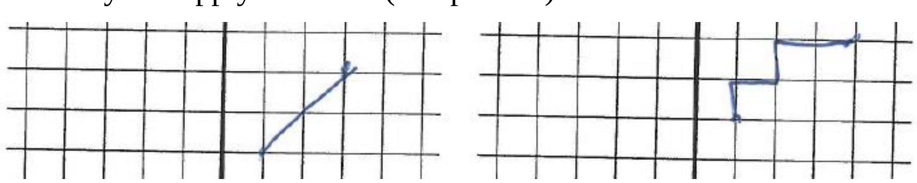
Figure 2: Nicole's illustrations of Euclidean and Taxicab distance, respectively.

Algebraic Representation of Distance

Nicole defined *distance* as "the measurement in between…two or more points that someone would ask me." Further, Nicole clearly explained that with the two geometries, "the definition [of distance] would be the same, but how to find it with the equation won't be the same," indicating that she distinguishes between these metrics. This is evidence of a process conception of *algebraic representation of distance* since she seemed to have a general definition for distance and would be able to find the distance between any two points in either geometry.

Geometric Representation of Circle

Nicole demonstrated and stated, that she is able to construct a Euclidean circle easily without plotting specific points on the circle (see Figure 3 - red ink indicates her drawing during the interview discussion). Developmentally, with the addition of Taxicab distance, Nicole reorganized her Euclidean Circle Schema to accommodate this new metric in order to describe and draw the Taxicab circle. When describing this circle in comparison to a Euclidean circle, she says "when I think of in Taxicab geometry …visually it won't be the same, but I do think the definition [of it] would be the same, because it has to be equidistant to be a circle." She explained that when she drew her Taxicab circle, she had to "follow strict routes, making my radius." She attempted to apply her written definition of circle by using the property of equidistance, and constructed her Taxicab circle incorrectly (see blue square in Figure 3, sketched during her individual work). This may be due to her inability to coordinate her *geometric representation of distance* in Taxicab geometry and her definition of circle as a set of equidistant points from a fixed point. More specifically, she was not able to imagine traveling a given distance in all directions from the center following 'strict routes.'

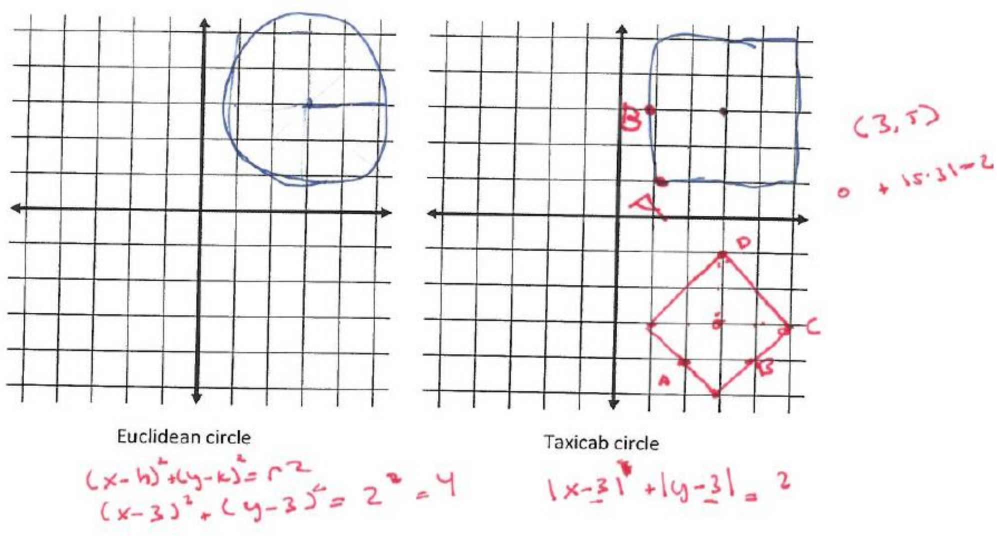

Figure 3: Nicole's illustrations of Euclidean and Taxicab circles, respectively.

During the interview, with prompting, Nicole recalled the shape of a circle in Taxicab geometry is a "diamond." She constructed this by finding four points (vertices) on the diamond and connected them (see red diamond in Figure 3), saying "oh, but it was like this". Thus, Nicole exhibited an action conception of *geometric representation of circle* in Taxicab geometry, since she relied on her memory for the shape of a Taxicab circle and needed external cues to draw it.

Deriving the Algebraic Representation of Circle

Nicole, with prompting, could recall the equation for the Euclidean circle, and arrived at the correct equation (see Figure 3). She seemed to be completing a sequence of steps, each provoked by the previous, since she first needed to write the general equation for a Euclidean circle, then identified the variables that would be replaced by the given center and radius, and finally plugged them in. By working off memory, Nicole exhibited an action conception of *algebraic representation of circle* in Euclidean geometry. Later, Nicole recognized some relationship between the formula for distance and the equation for a circle when prompted to derive the equation for a circle in Taxicab geometry. She stated, "I'm wanting to use…absolute values simply because we use absolute values for the distance? But that could be wrong." These statements imply Nicole saw the Euclidean distance formula is used in the Euclidean circle equation, and thus inferred the same must be true in Taxicab geometry. It appears she relied on this pattern to create her equation for the Taxicab circle. Thus, we believe that Nicole has an action conception of *algebraic representation of circle*.

Discussion and Concluding Remarks

Fischbein (1993) explains that in geometrical reasoning, a major obstacle is the tendency to "neglect the definition under the pressure of figural constraints," (p. 155). The results presented in this paper support this notion, with Nicole exhibiting a slightly different path to derive her Taxicab circle equation other than what our preliminary genetic decomposition illustrated. Our data indicates that Nicole had an action conception of *geometric representation of circle* and an object conception of *geometric representation of distance*, which allowed her to eventually draw the given Taxicab circle.

We expected that to derive the equation for a Taxicab circle, Nicole would need to have an object conception of *algebraic representation of distance.* Although she eventually arrived at the correct equation for her Taxicab circle with a process conception of *algebraic representation of distance*, her success was reliant on reproducing patterns instead of logic. Further, we claimed that she must have an object conception of a definition in Euclidean geometry to consider applying it in a new geometry. Nicole demonstrated that out of the concepts considered, she had an object conception of *geometric representation of distance* only, which could be why she struggled to derive the Taxicab circle equation from logic. Vinner (1991) and many others support this, since knowing a definition does not imply a real understanding of the concept.

As evidenced in this study, Nicole knew the definitions of *circle* and *distance,* but was unable to apply them effectively to derive an equation, and instead relied on patterns. We still believe an object conception of definitions is necessary prior to operating in another geometry to logically derive this equation. We plan, for future analysis, to identify the cognitive paths for each student and further investigate misconceptions these students illustrate in their discussion about deriving the equation for a Taxicab circle.

Questions for the Audience

1. What obstacles have your students faced when transferring their knowledge of definitions to a new context?
2. What are some good activities that can assist students in encapsulating their definitions?

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