Conceptual Analysis in Cognitive Research: Purpose, Uses, and the Need for Clarity

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This theoretical paper discusses conceptual analysis of mathematical ideas relative to its place within cognitive learning theories and research studies. In particular, I highlight specific ways mathematics education research uses conceptual analysis and discuss the implications of these uses for interpreting and leveraging results to produce empirically tested learning trajectories.

Keywords: Conceptual Analysis, Cognitive Research, Hypothetical Learning Trajectories

Cobb (2007) argued that mathematics education "can be productively viewed as a design science, the collective mission of which involves developing, testing, and revising conjectured designs for supporting envisioned learning processes" (p. 7). This requires that researchers' work leverages scientific methods to inform design (at the instructional, curricular, or institutional level) that positively impacts student learning.

Thus, a useful way to characterize cognitively-oriented research goals is the production of empirically tested learning trajectories that provide opportunities for students to construct ideal ways of reasoning about mathematical ideas within a coherent trajectory spanning their entire mathematical careers.[1] *Conceptual analysis* (Thompson, 2008a) plays an important role in this work, yet researchers are not always explicit about how they use conceptual analysis, nor are they clear about how conceptual analysis of an idea contributes to both the design and refinement of interventions that contribute to the broader goal of advancing knowledge in the field.

In this paper I will discuss conceptual analysis of mathematical ideas relative to its place within cognitive learning theories, highlight different ways that conceptual analysis is used in specific research studies, and explore how these uses contribute in different ways to achieving the overall goals of cognitively-oriented mathematics education research.

The Importance of Theory in Mathematics Education Research

Conceptual analysis focuses on defining mental activity characterizing both real and epistemic individuals' meanings, and as such derives from general constructivist principles. diSessa and Cobb (2004) and Thompson (2002) both describe theoretical perspective hierarchies starting from broader background theories like Piaget's (1971) genetic epistemology to more narrow domain-specific theories that "entail the conceptual analysis of a significant disciplinary idea…with the specification of both successive patterns of reasoning and the means of supporting their emergence" (diSessa & Cobb, 2004, p. 83). Background theories serve "to constrain the types of explanations we give, to frame our conceptions of what needs explaining, and to filter what may be taken as a legitimate problem" (Thompson, 2002, p. 192). Domain-specific theories address "ways of thinking, believing, imagining, and interacting that might be propitious for students' and teachers' mathematical development" (p. 194).

That conceptual analysis originated from radical constructivism has implications for its character and purpose. A description of what it means to understand a mathematical idea should be phrased in terms that reflect a researcher's epistemology, and not in a faint or elusive way. This is why conceptual analysis, as defined by Glasersfeld (1995), Steffe & Thompson (2000),

[1] The use of "ideal" here is framed as a goal even if consensus is never reached. Refinements to improve the effectiveness and coherence of students' mathematical experiences is the manifestation of this goal in practice.

and Thompson (2008a), is a description of cognitive states and processes. Grounding conceptual analysis in descriptions of mental actions and schemes attunes us to focusing on important ways of understanding foundational ideas that influence students' abilities to construct and leverage productive images of sophisticated ideas articulated by a researcher's learning goals and *hypothetical learning trajectory* (Simon, 1995).

Conceptual Analysis, Hypothetical Learning Trajectories, and Teaching Experiments

Thompson (2008a) defined conceptual analysis as a description of "what students must understand when they know a particular idea in various ways" (p. 42) and outlined four uses: 1) to build models of students' thinking by analyzing observable behaviors, 2) to outline ways of knowing potentially beneficial for students' mathematical development, 3) to outline potentially problematic ways of knowing particular ideas, and 4) to analyze coherence in meanings among some set of ways of knowing. From a Piagetian-constructivist perspective, understandings are organizations of mental actions, images, and conceptual operations. Describing an understanding—either actual or intended—therefore involves specifying the mental actions, images, and operations that constitute it. Conceptual analysis provides clarity on the mental actions that characterize particular understandings, their potential origins, and their implications for subsequent mathematical learning. Conceptual analysis does not produce a list of mathematical facts or particular learning objectives. Conceptual analysis is about articulating the cognitive processes that characterize particular understandings, which serves as a basis for task design and shapes researchers' identification of students' mathematical thinking and learning. Thus, conceptual analysis is a form of theory itself—an operationalization of what diSessa and Cobb (2004) call an *orienting framework* in the context of mathematics education research.

Ellis, Ozgur, Kulow, Dogan, & Amidon (2016) joined others (e.g., Clements & Samara, 2004; Sztajn, Confrey, Wilson, & Eddington, 2012) in stressing the importance of learning trajectory research. There is no consensus definition for hypothetical learning trajectory yet. Most descriptions are refinements of Simon's (1995) original definition as "[t]he consideration of the learning goal, the learning activities, and the thinking and learning in which students might engage" (p. 133). Hypothetical learning trajectories, as indicated by their name, should be framed as hypotheses to be tested in empirical studies, which often employ the teaching experiment methodology (Steffe & Thompson, 2000). As such, each of the three components of a hypothetical learning trajectory must be clearly articulated in enough detail so that during a teaching experiment, and in retrospect, it is possible for the researcher to provide empirical support for accepting or rejecting any part of the hypothesis. A teaching experiment, as described by Steffe and Thompson (2000), is the means by which to assess and refine hypothetical learning trajectories informed by a conceptual analysis. Teaching experiments have three parts, and different uses of conceptual analysis contribute to each part in different ways (see Figure 1).

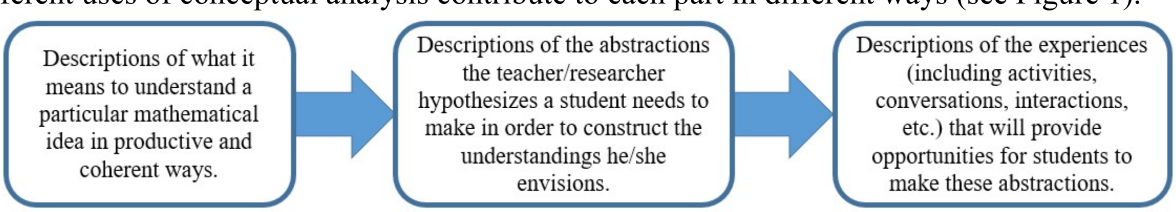

Figure 1. Parts of a teaching experiment.

Thinking in these terms, we can clarify how the results from different research studies contribute to the goal of creating empirically tested ideal mathematical learning trajectories.

Examples of Different Uses of Conceptual Analyses

Since researchers' contributions to learn trajectory research depend on how they used conceptual analysis, their conceptual analyses constitute an interpretive lens to make sense of their data and indicate the specific ways that others should leverage and interpret their work. The following three examples will help to illustrate this point. Each are drawn from compelling, influential research related to the teaching and learning of exponential growth.

Confrey and Smith's Retrospective Conceptual Analysis: Modeling Student Reasoning

Confrey (1994) and Confrey and Smith (1994, 1995) developed robust descriptions for students' images of multiplication, ratio, covariation, function, and rate based on retrospective conceptual analysis of teaching interviews. Student working through tasks like paper folding and predicting future values for an item retaining 90% of its value each year leveraged meanings for multiplication, rate of change, and function that often differed from conventional meanings. By carefully modeling students' schemes, Confrey and her colleagues described productive images that they claimed could be a powerful foundation for understanding exponential growth.

Images of multiplication, covariation, function, rate of change, and exponential growth. Confrey (1994) described thinking about multiplication via *splitting*. A *split* is the action of creating equal copies of an original amount or breaking an original amount into equal-sized parts. She then defined multiplication as the result of some *n*-split on an original whole and division as examining one of the equal parts of the split relative to the whole. Ratios rather than differences are then the natural means of comparison when conceptualizing splitting. Confrey and Smith (1994, 1995) described students engaging in *covariational reasoning* when coordinating splits and defined *covariation* discretely as a process of synchronizing successive values of two variables. A function relationship is then "the juxtaposition of two sequences, each of which is generated independently through a pattern of data values" (1995, p. 67) with specific function characteristics emerging from repeated actions during this coordination.

Confrey and Smith argued that students reasoning covariationally developed notions of *rate* that differed from conventional definitions. Some students coordinating arithmetic and geometric sequences to reason about exponential growth described the relationship as having a constant rate of change, meaning that thinking about rates as a ratio of additive differences is not an inevitable choice for students. They proposed defining *rate* in a way that respects students' intuitions. A *rate* is a unit per unit comparison where *unit* refers to what remains constant in a repeated action (Confrey, 1994). Thus, changes (and rates) can be conceived of additively or multiplicatively. Confrey and Smith argued that coordinating repeated addition to move through an arithmetic sequence with repeated multiplication to move through the geometric sequence and interpolating values by coordinating arithmetic means with geometric means is productive foundation for understanding exponential growth.

Commentary. Confrey and Smith's work modeled students' constructed schemes from empirical data and theorized about the utility of specific meanings for multiplication, covariation, function, and rate of change for understanding exponential growth. This kind of retrospective conceptual analysis is very useful for characterizing the way that some students productively reasoned about specific tasks spontaneously, including novel ways of thinking not typically emphasized in curricula. Confrey and Smith were not focused on generating detailed learning trajectories,[2] nor did they consider the implications for their specific meanings on understanding

[2] Weber (2002a, 2002b) and Ström (2008) both studied the implications of Confrey and Smith's conceptual analysis, as did Amy Ellis and her colleagues. I will say much more about Ellis et al.'s work later in this paper.

sophisticated mathematical ideas students will encounter in the future such as the Fundamental Theorem of Calculus (FTC). Their work was limited to modeling students' meanings for mathematical ideas within a fairly narrow scope of mathematical tasks and considering implications of these meanings for what they conceived as related ideas.

There are some limitations in studies using conceptual analysis in this manner, and understanding these limitations is critical to putting their results in perspective. In teaching interviews and experiments, results are always impacted by a researcher's choice of tasks and initial assumptions. For example, Confrey and Smith assumed that repeated multiplication is a useful foundation for defining exponential growth, and all of the tasks could be solved by (and perhaps encouraged) images of repeated multiplication. Since they were attuned to looking for productive ways of reasoning in these tasks, their conclusions depended on this initial assumption. Since results are influenced by the researchers' initial assumptions and task selection, their work does not compare the relative strengths of various potential meanings and learning trajectories. That requires a different use of conceptual analysis that looks more broadly at issues of coherence in mathematical ideas at all levels, which is not what Confrey and Smith sought to achieve. Scientific and mathematical progress throughout history is almost entirely a story about breakthroughs in understanding that defy human expectations and intuition. Thus, we should expect that classifying students' productive schemes for an idea will give us powerful insights into how individuals construct internally coherent schemes but not necessarily uncover ideal meanings we may want students to construct.

Thompson's Conceptual Analysis: Coherence of Mathematical Ideas Leading to Calculus

Thompson's (1994a) unpacking of the key ideas in calculus, particularly the FTC, motivated and informed his conceptual analysis for exponential growth (Thompson, 2008a). Thompson imagined a broadly coherent trajectory for students' mathematical experiences focused on quantitative reasoning, covariational reasoning, and representational equivalence that could unite most topics from grade school mathematics through calculus (Thompson, 2008b). Thus, his conceptual analysis considers exponential functions as just one of many opportunities for students to develop and apply particular ways of thinking.

Quantitative and covariational reasoning, rate of change, accumulation, and the FTC. Thompson's meanings for covariation, function, and rate of change are different from Confrey and Smith's because his goals are different. His work is grounded in *quantitative reasoning*, which describes conceptualizing a situation to form a quantitative structure that organizes relevant *quantities* (measureable attributes) and *quantitative operations* (new quantities representing a relationship between other quantities) (Thompson, 1988, 1990, 1993, 1994b, 2011, 2012). If someone sees a situation as composed of quantities that change together and attempts to coordinate their variation, then she is engaging in *covariational reasoning* (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). Sophisticated covariational reasoning involves linking two continuously varying quantities to create a *multiplicative object*, a unification that combines the attributes of both quantities simultaneously (Saldanha & Thompson, 1998; Thompson, 2011; Thompson & Carlson, 2017).

Thompson (1994a, 1994b) and Thompson and Thompson (1992) outline an image of constant rate as a proportional correspondence of two smoothly covarying quantities. When one quantity's magnitude changes by *any* amount, the other quantity's magnitude changes proportionally. This was Newton's image of rate that allowed him to conceptualize the relationship between accumulation and rate of change expressed formally in the FTC (Thompson, 1994a, 2008a). Over small intervals, he imagined that any two covarying quantities change together in a proportional correspondence. This can be modeled by a piecewise constant rate of change function and its corresponding piecewise linear accumulation function. The FTC describes how these two functions are related as the interval sizes tend to zero. See Figure 2.

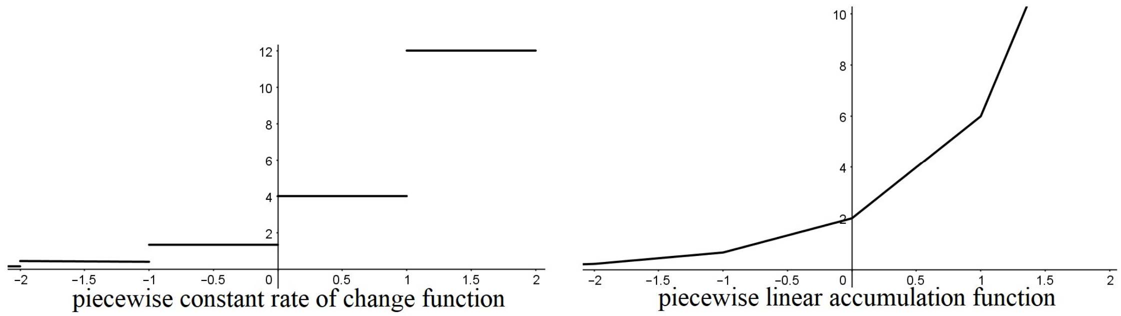

Figure 2. Piecewise linear accumulation function and piecewise constant rate of change function.

Exponential functions. Building from his images of constant rate of change and the FTC, Thompson's (2008a) conceptual analysis involved thinking about classifying functions based on similarities in their rate of change functions and imagining a function as emerging through accumulation. Specific to exponential functions, he conceptualized a relationship with a rate of change on some interval that is always proportional to the function value at the beginning of the interval. As the interval size decreases, the piecewise linear accumulation function converges to an exponential function. Thompson (1994a, 2008a) argued that this way of understanding allows a person to conceptualize both change and accumulation as happening simultaneously, makes it natural to imagine the function value growing continuously and producing outputs for all real number inputs, is consistent with a coherent way of reasoning about all function relationships, and leads to a productive operational understanding of the FTC.

Commentary. Much like Confrey and Smith, Thompson's work is not a detailed hypothetical learning trajectory.[3] Thompson's conceptual analysis is part of a broader, idealized web of ideas stretching from students' first mathematical experiences through calculus. It does not consider students' actual mathematical background experiences in modern classrooms, the cognitive load it places on students, or whether the ideas reasonably coincide with common ways students may attempt to spontaneously reason about tasks. It also depends on a different meaning for function relationships, how functions are categorized, and the foundational criterion for a relationship to be exponential. In Thompson's conceptual analysis, exponential growth is related to repeated multiplication almost by coincidence and is not the foundational meaning.

Ellis and Colleagues: From Exploratory to Hypothetical Learning Trajectory

Ellis and her colleagues (Ellis, Ozgur, Kulow, Williams, & Amidon, 2012, 2015; Ellis et al., 2016) mostly leveraged Confrey and Smith's images of covariation, rate, and exponential growth to construct a rough exploratory learning trajectory surrounding a single context. Ellis et al. extended and clarified how Confrey and Smith's ideas might productively support students' understanding of exponential relationships and chose a situation where they conjectured students could easily justify that the function's domain and range were not restricted to a set of discrete values. They built a Geogebra applet showing the image of a plant (the Jactus) with a height that varied exponentially with elapsed time. The applet's user can vary the elapsed time by sliding the plant along the horizontal axis and its height would update in real time. The applet also displays the time elapsed and the plant's height as an ordered pair as the user slides the plant horizontally.

[3] Castillo-Garsow (2010) did produce a learning trajectory and empirical study based on this conceptual analysis.

In designing their study, Ellis et al. anticipated, and later confirmed, that students' initial models for exponentiation involved an informal image of repeated multiplication. Ellis et al. wanted students to leverage covariational reasoning to build a more robust image of exponential growth focused on coordinating multiplicative changes in one quantity with additive changes in another quantity. With this understanding, students might understand b^x as both the possible height of a plant at some moment in time and as representation of a (multiplicative) change in height. Students working through the activities exhibited key shifts in their thinking reflecting increased attention to how the two quantities changed together over intervals of varying size. "[These] results…offer a proof of concept that even with their relative lack of algebraic sophistication, middle school students can engage in an impressive degree of coordination of co-varying quantities when exploring exponential growth" (Ellis et al., 2012, p. 110).

Commentary. Ellis et al. used conceptual analysis in three ways. First, they further unpacked Confrey and Smith's conceptual analysis of exponential growth as students might construct it from images of coordinating additive and multiplicative changes. Second, they continuously modified and updated their exploratory learning trajectory and tasks throughout the study based on models of students' schemes. These analyses, coupled with retrospective analysis on the empirical data, allowed them to craft highly detailed descriptions of students' meanings at various points in time and how those meanings developed through interactions with tasks and teaching interventions (Ellis et al., 2016). The result is the foundation for a powerful hypothetical learning trajectory. Ellis et al. now have empirical grounding for theories on how students may come to construct specific meanings related to exponential growth and related ideas. The refinements from the exploratory research and their model for how students construct specific meanings in specific contexts is now a fully realized hypothesis for systematic testing.

Ellis et al.'s work is an impressive example of critical work in developing empirically tested learning trajectories and demonstrates how initial exploratory work in developing an understanding of students' scheme construction, like the work of Confrey and Smith, can be refined and expanded to contribute to important work on learning trajectories. However, as they note, "Our learning trajectory is an attempt to characterize the nature of the evolution of students' thinking in a particular instructional setting" (2016, p. 153) and is thus only one of many possible learning trajectories. Like Confrey and Smith, their work assumes that repeated multiplication is the starting point from which to develop an understanding of exponential growth. In fact, the initial activities in their exploratory learning trajectory encouraged and then attempted to modify this reasoning. Their work does not extend to considering the long-term implications for students who develop their intended meanings compared to students with other potential meanings for exponential growth, nor does it (as of yet) seek to explain persistent challenges students encountered. This was not the role of the described study but does describe critical future research.

Summary and Theoretical Implications

A teaching experiment is a method of testing a research hypothesis (a carefully detailed hypothetical learning trajectory) informed by conceptual analysis that analyzes the degree to which (and aspects of) tasks and interactions that promoted specific abstractions. None of the research studies described in this paper satisfy these criteria of a formal teaching experiment because the empirical work, when present, was more exploratory in nature. However, each of them contribute to the goals of cognitively-oriented mathematics education research in powerful ways. Confrey and Smith described students' schemes related to repeated multiplication based on spontaneous reasoning about particular mathematical tasks. Ellis et al. further unpacked these schemes and, based on retrospective analysis of empirical data, produced a well-defined hypothetical learning trajectory for specific meanings using specific tasks that now has the clarity and specificity necessary to be a scientific hypothesis. Thompson's work takes a broader view and suggests ways of understanding exponential growth situated within a coherent body of mathematical ideas extending beyond a single topic.

Currently there is no consensus on the exact meaning of a hypothetical learning trajectory. Ellis et al. (2016) have an excellent literature review detailing the different interpretations. In addition, reflecting on their work suggests that the field may benefit from greater clarity in defining different types of learning trajectories with the definitions influenced by the role of conceptual analysis. A potential starting point is given below.

- *Exploratory learning trajectory* – Conceptual analysis (either based on a researcher's analysis of mathematical ideas or based on empirical data) can suggest potentially useful ways of understanding particular ideas. A researcher then creates tasks and a rough exploratory trajectory for gathering empirical data on how students reason about specific contexts in specific settings. Since the enacted learning trajectory is continually modified based on modeling students' emerging meanings, this is not yet a scientific hypothesis.
- *Enacted learning trajectory* – An actual learning trajectory unfolded based on the exploratory learning trajectory. Conceptual analysis is used retrospectively to describe how students' schemes changed as a result of their mathematical activity.
- *Hypothetical learning trajectory* – This describes a specifically stated research hypothesis outlining specific targeted mental actions and schemes, specific tasks and a task sequence, and descriptions of how those tasks will contribute to students accommodating their schemes. The teaching experiment that tests this hypothetical learning trajectory seeks to accept or reject particular aspects of the hypothesis, and will ultimately result in refinement. Conceptual analysis is critical to the design of the learning trajectory and retrospectively in analyzing outcomes in the more formal teaching experiment.
- *Empirically supported learning trajectory* – After potentially several rounds of refinement and testing with hypothetical learning trajectories, a researcher can articulate an empirically supported learning trajectory. In comparing the results and implications of competing empirically supported learning trajectories, researchers can move closer to a learning trajectory that supports the development of ideal ways of understanding.

Any of these learning trajectories could be narrow in scope (focused on a particular mathematical idea) or grand in scope (focused on students' learning as an arc from grade school through graduate level mathematics). Researchers' questions of interest and how they use conceptual analysis dictate the type of learning trajectory they are developing and studying, and the scope of their work dictates their contribution to the field from models of students' schemes relative to particular ideas to coherent mathematical experiences across many topics and grade levels.

As researchers, we are obligated to not only produce scientifically-valid findings but also to communicate our work in ways that allow others to leverage our results to advance the collective mission of our design science. Being more explicit about the role of conceptual analysis in our work and having greater clarity on how our learning trajectory research contributes to design research can help us achieve this. I hope that my articulation of how different uses of conceptual analysis are relevant to developing different kinds of learning trajectories facilitates relevant and productive communication among cognitively-oriented, qualitative mathematics education researchers.

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