Framework for Students' Understanding of Mathematical Norms and Normalization

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Mathematical norms and normalization of vectors are important concepts used throughout the mathematical and physical sciences; however, very little research has been done on students' understanding of these concepts. To remedy this lacuna, this report presents a framework that can be used to model, explain, and predict the ways students reason about and solve problems involving norms and normalization.

Keywords: norms, normalization, student understanding, vectors, linear algebra

Normalization of particular vectors from various vector spaces (e.g., $\mathbb{R}^n$, $\mathbb{C}^n$, function spaces) is mathematically important in various contexts. Some examples include directional derivatives in multivariable calculus, states of quantum mechanical systems in Physics, and the development of orthonormal bases through the Gram-Schmidt process in Linear Algebra and Numerical Analysis. Despite the wide applicability of normalization within mathematics and science, students' understanding of norms and normalization seems not to have been studied. Research that has examined students' understanding of absolute value have come close to the topics (e.g., Almog & Ilany, 2012; Sierpinska, Bobos, & Pruncut, 2011), but have not directly addressed them. This lack of research into students' understanding of norms and normalization must be remedied.

In this report, I present a framework for students' understanding about mathematical norms and normalization. This framework aims to address the following research questions: (a) What are the various components involved in understanding mathematical norms and normalization, and how are they interconnected? and (b) How does a students' understanding of those components impact their thinking and solution strategies when working on problems involving norms and normalization?

I first explain the theoretical lens of the Emergent Perspective that I adopt within this report, and why the development of models of student thinking and understanding are important. Next, I describe the methods for the study and framework development. An explanation of the framework is given afterwards, with a focus on how the various components fit together and interact with one another. I then use data from two students to illustrate how the framework can be used to model, explain, and predict students' reasoning about normalization problems. Lastly, I discuss how the results illustrate elements of understanding norms and normalization that were particularly powerful for students in their reasoning about normalization problems, and the implications these have for teaching and future research.

Theoretical Lens

The Emergent Perspective (Cobb & Yackel, 1996) coordinates psychological constructivism (von Glasersfeld, 1984, 1995) and social interactionism (Bauersfeld, Krummheuer, & Voigt, 1988) into a version of social constructivism that views mathematical learning as both individual construction and enculturation into the mathematical community. As an elaboration on the Emergent Perspective, Rasmussen, Wawro, and Zandieh (2015) added the importance of understanding the conceptions individual students bring to bear in their mathematical work. The main goal of this research is to gain a better understanding of students' conceptions about norms and normalization, and create a framework for modeling, explaining, and predicting students' reasoning about these concepts.

Methods and Framework Development

The framework developed and used herein was inspired and influenced by Zandieh's (2000) framework for student understanding of derivatives and Lockwood's (2013) model of students' combinatorial thinking. Similar to the work of Lockwood (2013), I used a *conceptual analysis* (von Glasersfeld, 1995) or "a detailed description of what is involved in knowing a particular (mathematical) concept" (Lockwood, 2013, p. 252) to create this framework of students' understanding of norms and normalization. This conceptual analysis involved an iterative process of moving among my own theoretical thinking about the constructs involved in understanding norms and normalization, relevant literature, and the student interview data.

Although research examining students' understanding of norms and normalization is scarce, research on students' understanding of absolute value is relevant, as the absolute value is an example of a norm. Important findings include: the power in understanding multiple ways to define the absolute value (e.g., delete the negative sign, $|x| = \sqrt{x^2}$) in solving different problems (Wilhelmi, Godino, & Lacasta, 2007); and the power of understanding absolute value as the magnitude of a number or its distance from zero (Almog & Ilany, 2012; Sierpinska, Bobos, & Pruncut, 2011). These ideas impacted the development of the framework, and may be important for students' understanding of norms and normalization, as I illustrate later.

The data used in the development of the framework consists of hour-long, video-recorded, semi-structured interviews with individual students at two different collection sites: nine junior-level quantum mechanics students from a university in the northwestern United States; and two junior-level linear algebra students and two sophomore-level multivariable calculus students from a university in the southeastern United States. Students at the first site were asked questions about several linear algebra concepts including normalization, while students at the second site were only asked questions about norms and normalization. Although interviews from both sites informed the framework development, the data used within this report to illustrate the utility of the framework come from the second collection site.

In analyzing the student data, I first watched the sections of the interview in which students explained their understanding of normalization and normalized vectors from $\mathbb{R}^2$ and $\mathbb{C}^2$, writing a summary of each student's thoughts afterwards. Next, the transcript or video of the interview was coded (Maxwell, 2013) for each student, with some codes influenced by the state of the framework at the time of coding. Lastly, I examined the framework to see how well it could be used to model and make sense of each student's thinking and reasoning about norms and normalization, making modifications as necessary. The framework, as it stands in the next section, has gone through several revisions and refinements based on this analysis, as well as feedback received through poster presentations at two conferences (Watson, 2017a, 2017b).

Framework for Students' Understanding of Normalization

Figure 1 presents a visual representation of the framework. I contend that understanding normalization essentially involves three major components, namely the norm of a vector, procedures for normalizing a vector, and what a normalized vector is (as conveyed by the three large ellipses in Figure 1). I expand on the contents of these ellipses in the following subsections. The lack of directional arrows in the figure is deliberate, as any component could inform how a student thinks about any of the other components, although when normalizing a vector, students

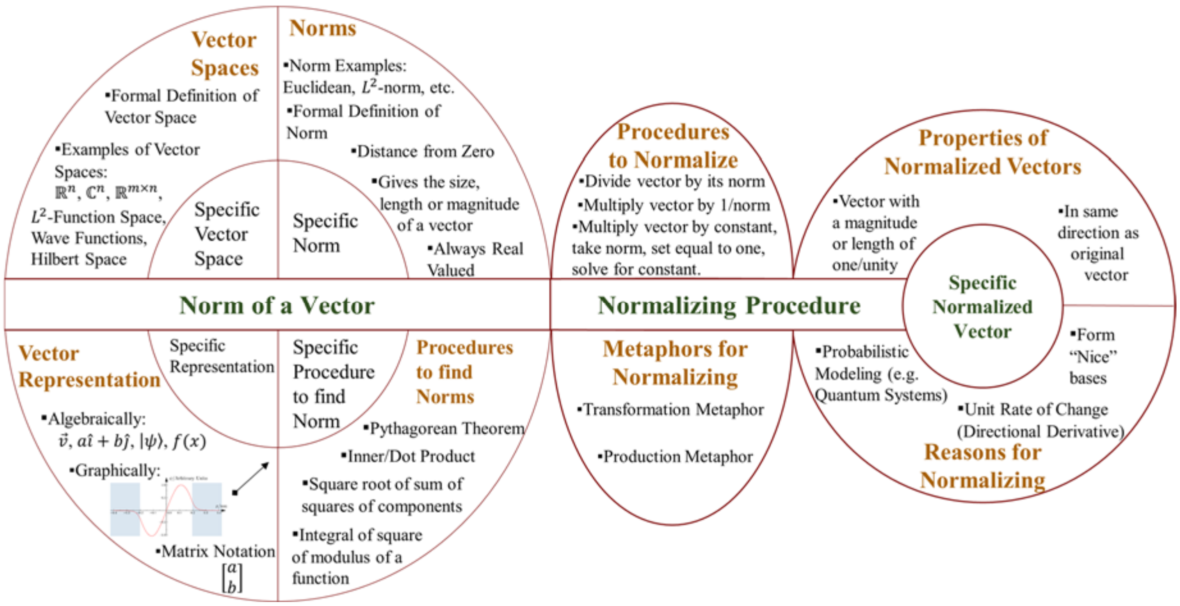

Figure 1: Framework for students' understanding of norms and normalization

generally find the norm, perform a normalizing procedure, and end with a normalized vector (i.e., left to right in the figure). Finally, students' understandings of norms and normalization do not necessarily include all of these components and connections; as such, when using the framework to model a student's understanding, components and connections presented in Figure 1 could be scarcer or even missing for a particular student's model.

Norm of a Vector

I have found four elements that can influence or determine how a student finds the norm of a vector, namely the vector space the vector is an element of, the representation chosen for the vector, the particular norm function to be used, and the procedure chosen for finding that norm (these four aspects are represented by the inner ellipse on the left of Figure 1). Additionally, a student's broader or more general understanding of vector spaces, representations of vectors, norms, and different procedures for finding norms, discussed in further detail below, can also inform and influence how the student finds the norm of a specific vector

A student's understanding of vector spaces could include examples of several vector spaces, such as $\mathbb{R}^n$, $\mathbb{C}^n$, or L^2-function space, although many students only have experience with vectors in $\mathbb{R}^n$. Mathematically sophisticated students may also be able to draw on their understanding of the formal definition of vector space. Altogether these can influence how a student thinks about a specific vector, which can inform their normalization of it.

A student's understanding of vector representations could include examples of algebraic notations (e.g., letter with special marking, functions, Dirac Notation), graphical notations (e.g., graphs of functions, points on a Cartesian coordinate system, directional arrows), and matrix notation (i.e., column or row vectors), with each representation choice affecting how a student thinks about finding the norm of a vector. Furthermore, a student's understanding of why we use representations, and their ability to select the best or most useful representation for a given task—which is part of Meta-Representational Competence (diSessa, Hammer, Sherin, & Kolpakowski, 1991; diSessa, 2004; Wawro, Watson, & Christensen, 2017)—could also impact a student's thinking about norms and normalization.

For many undergraduate students, the only norm they are explicitly aware of is the Euclidean Norm on $\mathbb{R}^n$, as most have only heard the term "norm" in conjunction with real vectors. However, mathematically sophisticated students may also know examples of other norms, the formal definition of norm, and important properties of norms (e.g., always real valued). Any of these ideas about norms can shape how a student finds the norm of or normalizes a vector.

There is also great variety in how students approach finding the norm of a vector for a given norm. For instance, a few ways students can find the Euclidean norm of a vector in $\mathbb{R}^2$ are to take the square root of the sum of the squares of the components, take the square root of the dot product of the vector with itself, or graph the vector and use the Pythagorean Theorem to find the length. While all of these are correct, each procedure can influence how students think about the norm, and their understanding of multiple procedures and connections between them could be drawn upon at any time.

Normalizing Procedure

There are several different ways a student can normalize a vector, such as dividing the vector by its norm, multiplying the vector by the reciprocal of its norm, or multiplying the vector by an unknown constant *before* finding the norm, setting it equal to one, and solving for this normalization constant. Moreover, there seem to be essentially two metaphorical expressions (Zandieh, Ellis, Rasmussen, 2017) students call upon when normalizing a vector which influence how they think about, and even notate, the normalized vector. The *transformation/morphing metaphor* views normalizing as a procedure that transforms or morphs the original vector into the normalized one, as when a student talks about "shrinking" the original vector down to a length of one. The *production metaphor*, on the other hand, views normalizing as a procedure that produces a vector that is in the "same direction" as the original vector, but has a length of one.

Normalized Vector

A student's understanding of normalized vectors includes ideas about properties of normalized vectors and reasons why normalization is important. The properties could include normalized vectors having a norm, length, or magnitude of one, and being in the same direction as the original vector. Reasons for normalization students have in their understanding could include probabilistic modeling (such as in quantum mechanics), looking at unit rates of change (such as with directional derivatives in multivariable calculus), or even simply a rule or procedure that must be carried out as a part of some algorithm.

Using the Framework to Model Students' Understanding of Norms and Normalization

I now demonstrate how the framework can be used to model students' understanding using data from two students, Luke and Spencer. Luke was a physics/mathematics double major who came from an advanced linear algebra course, and Spencer was a computer engineering major who came from an introductory multivariable calculus course at the time they were interviewed. The interview consisted of having them solve three problems related to norms and normalization (an absolute value problem $|x-7|=3$; normalizing a vector in $\mathbb{R}^2$; normalizing a vector in $\mathbb{C}^2$ with components 3 and $3i$), and explain their own understanding of absolute value, norms, and normalization in general.

Although Spencer did briefly mention that the absolute value can tell you how far away a number is from zero, he struggled to make use of this fact in solving problems, and did not see how absolute value could be related to mathematical norms. In fact, Spencer always described a process for normalizing vectors when asked about norms, and essentially seeing “norm” and “normalize” as the same idea:

*I*nterviewer: Do you see a difference between norm and normalization, or are those just, like, so related that...?

Spencer: I mean, I see 'em pretty related. Um, I don't really see a difference between them, honestly.

Furthermore, what Spencer understood by normalization seemed particularly narrow, as evidenced in his work on normalizing the vector $\boldsymbol{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$. Before even starting the problem, he changed the vector $\boldsymbol{v}$ to the form $2\hat{\imath} + 5\hat{\jmath}$, and proceeded by finding the square root of the sum of the components, $\sqrt{2^2 + 5^2}$, to get $\sqrt{29}$. He then explained that he would need to divide by $\sqrt{29}$ to get one, and proceeded to divide each component by $\sqrt{29}$ to arrive at his solution of $\frac{2}{\sqrt{29}}\hat{\imath} + \frac{5}{\sqrt{29}}\hat{\jmath} = 1$. I then asked him what normalization means to him, and why he chose that particular procedure:

Spencer: Normalize to me is basically making, like, getting the, getting this one, basically. … Like, so, when you use the distance formula, obviously, like, this is the distance formula [pointing at the square root of the sum of the squares]. But, um, you want, like, the distance to be [pause]. You want it to be one. … Um, like, but other than that, I never, I've never actually asked why?

Interviewer: Why did you choose that procedure to normalize?

Spencer: It's really the only one that I know. And it's the most recent one in my head. We just went over it, I think, last week. So, that's, that's the most recent one. And, I think that's the only one that I know as of now. I think that's it.

Spencer did seem to realize that the distance formula is an integral part of normalization, but the discussion above continues to affirm that Spencer did not have a strong understanding of norm, and rather understood norm and normalization as being one in the same, namely a procedure he was taught to carry out. In fact, when he tried to make sense of normalization graphically, he plotted the original vector and normalized vector as points on a Cartesian plane, but was never confidently able to describe the relationship between the two points, or what it meant for the normalized vector to have a magnitude of one.

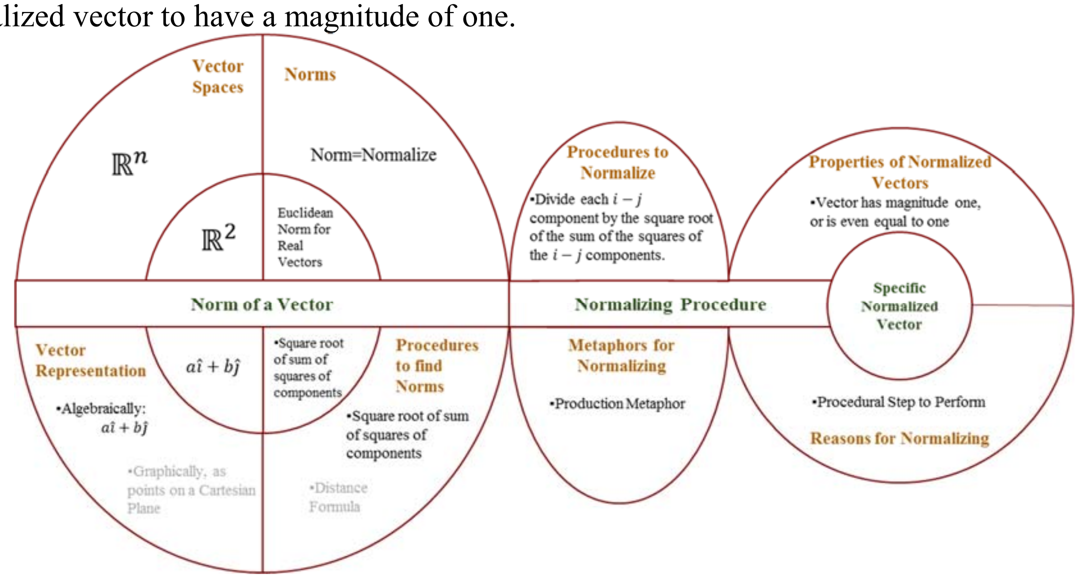

Figure 2: Model of Spencer’s understanding of normalization

Based on the above discussions with Spencer, I present a model of his understanding of normalization in Figure 2, which uses the framework as an organizational tool. This model can help us make sense of Spencer's understanding, but also gives us power to predict how he might approach normalizing vectors from unfamiliar vector spaces. More specifically, we would predict that Spencer would rely on the only procedure he knows for normalization, and probably encounter moments of uncertainty and confusion in the process.

As predicted by the model, when Spencer was asked to normalize the vector $\begin{bmatrix} 3 \\ 3i \end{bmatrix}$, he again attempted to write the vector as $3\hat{\imath} + 3i\hat{\jmath}$, which alone bothered Spencer, having the imaginary i and the normalized basis vector $\hat{\imath}$ in such close proximity. He then used his method of finding the square root of the sum of the squares of the components, and arrived at a value of zero.

Spencer: So, um, [pause]. I don't think it can be normalized? 'Cause I got zero.

Interviewer: OK, what do you mean by...?

Spencer: … you can't really divide anything by zero. … I just don't think it can be normalized at that point.

Spencer's approach to solve this problem could be modeled by replacing the $\mathbb{R}^2$ in the model above with $\mathbb{C}^2$, with all other components essentially the same. As soon as he got zero, however, his procedure broke down, leading him to declare the vector as something that cannot be normalized. Spencer was not able to draw on any generalized understandings of norms or normalized vectors to help him rethink his procedure for normalization. Still, by the end of the interview Spencer realized his understanding of normalization was limited, and he even mentioned the possible existence of other types of normalization.

To see a stronger understanding of norms and normalization, I present my model for Luke's understanding in Figure 3, and highlight important aspects of it. First, Luke had a strong understanding of absolute value as representing a number's distance from zero, and

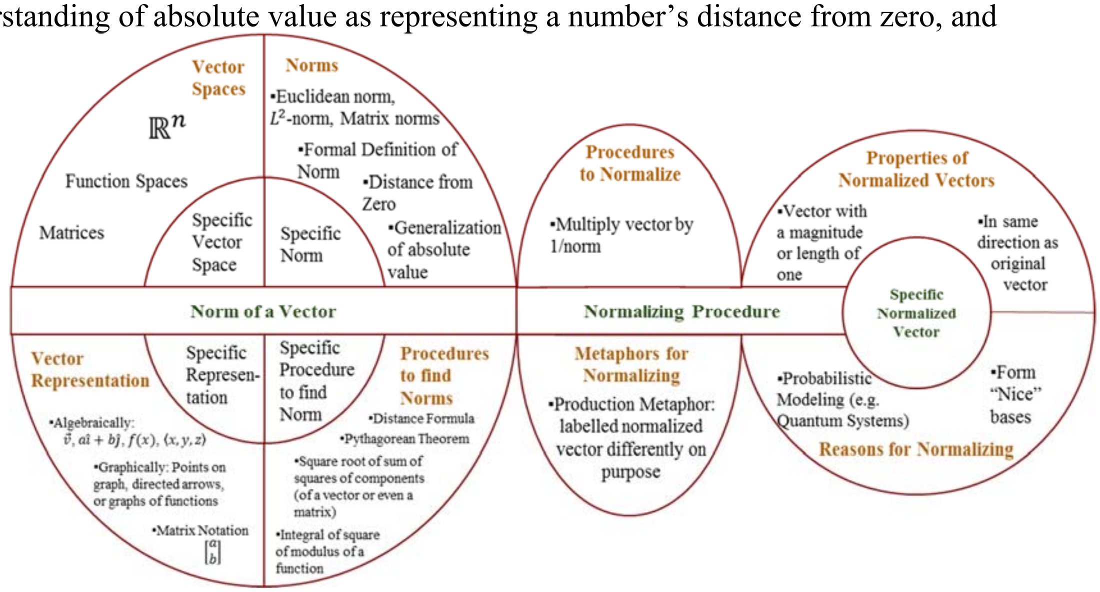

Figure 3: Model of Luke's understanding of normalization

understood norms as a generalization of this idea, namely giving the distance a vector is from the zero vector. Second, this distance from zero conceptualization was not limited to $\mathbb{R}^n$, as Luke was able to describe what the norm would be for several vector spaces, and how this could be thought of as giving a distance from zero. As an example, consider his description of a possible norm for the vector space of matrices:

Luke: So, if you have a matrix ... that would be, like, you know, you could define the norm where it's, like, just the square of the sum of all of 'em [the entries of the matrix]. Or, whatever you wanted to do. And that would be the distance from the zero matrix in the same sort of way, because, the zero matrix would be all zeros.

Third, Luke was confidently able to use and move among multiple vector representations and procedures for finding norms. And fourth, Luke understood the result of normalization, and why it is important for the creation of orthonormal bases, as well as probabilistic modeling in quantum mechanics.

Based on Luke's model, we would predict Luke to successfully draw upon his strong understanding to make sense of normalizing vectors from vector spaces that he was not familiar with. This was evidenced in his work to normalize the vector $\begin{bmatrix} 3 \\ 3i \end{bmatrix}$. Luke was somewhat unsure of how to proceed with normalizing this vector, and his first attempt was similar to Spencer's above, arriving at a norm of zero. However, unlike Spencer, Luke was able to draw on his understanding of norms, and immediately recognized this could not be correct:

Luke: So, I did something wrong, 'cause that's not the zero vector.

Interviewer: So, why did you think you did something wrong?

Luke: Well, the way a norm is defined, says that like, the norm of a vector can only be zero if that vector is zero. And, these entries are not zero. [I: OK]. So, there's something wrong with how I've been doing this. And, I bet, if I just took the modulus of each one first, then it would work.

Even more striking is the fact that Luke was able to propose a modification (taking the modulus of both components first) that was viable and mathematically sound, and he went on to correctly normalize this complex vector.

Discussion and Conclusion

Luke's understanding of norm representing a distance from zero was particularly powerful for making sense of norms in multiple vector spaces, even vector spaces that were unfamiliar to him. This coincides with Sierpinska et al. (2011) who explained the power in understanding absolute value as a distance from zero:

> Definitions based on the notion of distance are important in applications and in mathematical theory, in particular in generalizations of absolute value to norms in higher dimensions and general vector spaces, and in generalizations of limits and continuity in topology. (p. 280)

This also relates to my own findings on the importance of understanding norms for a strong understanding of normalization, including its importance for multiple contexts within science and mathematics (Watson, 2017a, 2017b).

In this report I have presented a framework for students' understanding of mathematical norms and normalization. This framework has identified the components that go into understanding mathematical norms, normalization, and normalized vectors. Furthermore, I have shown how students' solution methods for mathematical problems involving these concepts can be thought about, made sense of, explained, and even predicted by using the framework to model students' understanding. It is hoped that this framework will be helpful for future research into students' understanding of norms and normalization, as well as their understanding of related concepts such as metrics and metric spaces within real analysis and topology. Furthermore, this framework could be used by instructors to think about ways they might best help their students develop robust understandings of norms and normalization within a variety of courses where these concepts are used, such as helping students conceptualize norm as a distance from zero.

References

Almog, N., & Ilany, B.-S. (2012). Absolute value inequalities: High school students' solutions and misconceptions. *Educational Studies in Mathematics, 81*, 347–364.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*(3/4), 175–190.

diSessa, A. (2004). Metarepresentation: Native competence and targets for instruction. *Cognition and Instruction, 22*(3), 293–331.

diSessa, A., Hammer, D., Sherin, B., & Kolpakowski, T. (1991). Inventing graphing: Meta-representational expertise in children. *Journal of Mathematical Behavior, 10*, 117-160.

Hillel, J. (2000). Modes of description and the problem of representation in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 191–207). Dordrecht, Netherlands: Kluwer.

Lockwood, E. (2013). A model of students' combinatorial thinking. *Journal of Mathematical Behavior, 32*, 251-265.

Maxwell, J. A. (2013). *Qualitative research design: An interactive approach* (3rd ed.). Thousand Oaks, CA: Sage.

Rasmussen, C., Wawro, M., & Zandieh, M. (2015). Examining individual and collective mathematical progress. *Educational Studies in Mathematics, 88*, 259–281.

Sierpinska, A., Bobos, G., & Pruncut, A. (2011). Teaching absolute value inequalities to mature students. *Educational Studies in Mathematics, 78*, 275–305.

von Glasersfeld, E. (1984). An introduction to radical constructivism. In P. Watzlawick (Ed.), *The invented reality* (pp. 17–40). New York: Norton.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. London: Falmer Press.

Watson, K. L. (2017a, July). *Framework for students' understanding of mathematical norms and normalization*. Poster presented at the 2017 Transforming Research in Undergraduate STEM Education (TRUSE) Conference, St. Paul, Minnesota.

Watson, K. L. (2017b, July). *Quantum mechanics students' understanding of mathematical norms and normalization*. Poster presented at the 2017 Physics Education Research Conference, Cincinnati, OH.

Wawro, M., Watson, K., & Christensen, W. (2017, February). *Meta-representational competence with linear algebra in quantum mechanics*. Paper presented at the 20th Annual Conference on Research on Undergraduate Mathematics Education.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education IV* (pp. 103-127). Providence, RI: American Mathematical Society.

Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A characterization of a unified notion of mathematical function: the case of high school function and linear transformation. *Educational Studies in Mathematics, 95*(1), 21-38. doi:10.1007/s10649-016-9737-0