Leveraging the Perceptual Ambiguity of Proof Scripts to Witness Students' Identities

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Recognizing identity not only as an important educational outcome but also being inter-related to students' knowledge and practice, this paper explores an affordance of proof scripts; the witnessing of students' identities. Drawing on proof scripts from teaching experiments and the construct of perceptual ambiguity, this paper argues that proof scripts afford access not only to students' understandings, problematics, and ways of reasoning but also students' identities.

Key words: Identity, perceptual ambiguity, proof scripts

There exists a host of reasons for why researchers have grown increasingly interested in identity. Yet, this interest has not been accompanied by a growth in methodologies that afford the study of identities. The aim of this paper, therefore, is to demonstrate how the perceptual ambiguity of proof scripts can be leveraged to explore students' identities. To accomplish this goal, three steps are taken. First, I explore why interest in identity has grown in recent decades. Second, I describe the proof script methodology. Third, I present a characterization of perceptual ambiguity and then draw on proof script data to illustrate and discuss how proof scripts afford access not only to students' understandings, problematics, and ways of reasoning but also their identities, if we work to leverage their perceptual ambiguity.

Identity

Students' identities have become increasingly of interest to researchers. As researchers have grown in their capacity to document student affect and its impact on learning (cf. Bishop, 2012), organizations (e.g., NCTM, 2000) have increasingly called for teachers and researchers to work to better understand the emergence of productive dispositions. Also, researchers interested in equity oriented instruction have increasingly recognized not only that knowledge and practice are interactively constituted but also that students' identities impact students' knowledge and practice. Boaler (2002), for instance, has argued that classroom learning is constituted through interactions between students' knowledge, identities, and practices (see Figure 1); arguing, like Wenger (1998), that learning "is an experience of identity" (p. 215). Last, Bishop (2012), has argued that due to their impact on dispositions, affect, persistence, and achievement, identities are recognized as an important educational outcome.

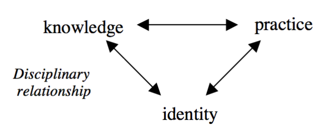

Figure 1. Boaler (2002) Learning Model

Despite agreement on its importance, definitions of identity vary widely. Due to length limitations, the discussion will focus on the definition of identity proposed by Bishop (2012). Bishop (2012) defines *identity* as: "a dynamic view of self, negotiated in a specific social context and informed by past history, events, personal narratives, experiences, routines, and ways of participating…. (it) is both individually and collectively defined" (p. 38). This definition highlights that identities are interactively constituted within environments and, in part, by others.

One of the primary means for interactive constitution is *discourse* – a point emphasized by Gee (2001, 2005) who argued that identities are created through discourses, by Sfard and Prusak (2005) who define one's identity in terms of internalized communications and narratives, and by Bishop (2012) who argues "discourse plays a critical role in enacting identities." Indeed, identities become visible through discourse as interlocutors position themselves and others in relation to their current social context, institutional setting, and history – a point poignantly illustrated by Setati (2005), who studied how class and power are enacted through mathematics classroom discourses. Like, Bishop (2012) and Setati (2005), the position taken in this paper is that discourses can serve as a primary means for exploring identity.

The Proof Script Methodology

Interest in students' reading strategies, difficulties with, and comprehension of mathematical proofs, has led to a host of studies. These studies have either employed proof comprehension assessments (*cf*. Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012), clinical studies (*e.g*., Weber, 2015), or novel methodologies, such as *proof scripts* (Koichu & R. Zazkis, 2013; D. Zazkis, 2014; R. Zazkis & D. Zazkis, 2016). The latter entails having students produce a written dialog where the interlocutors discuss a given proof, highlighting *problematics* (i.e., difficulties identified by students) and elaborating on key points to promote understanding. The dialogs are then analyzed by researchers to create models of students' understandings of the content and practices attended to, their perceptions of key points and ways of reasoning about *problematics*. This methodology emerged for a variety of reasons. First, as noted by Koichu and Zazkis (2013), past research examined students' difficulties from the researchers point of view. Methods were needed that enabled the identification of problematics, a point also emphasized by Brown (2017). Second, the methodology aligns with theoretical perspectives that see discourse as integral to thinking. Specifically, the methodology builds on Sfard's (2007) commognitive theory, in which Sfard argues thinking can be viewed as individualization of "the activity of communicating" (p. 571) that is derived from the collective patterned activities one experiences.

Perceptual ambiguity as a means to witness students' identities

At its core the proof script methodology calls on students to produce a written dialog in which the participants discuss a proof, paying special attention to the key ideas and problematics observed. Taking the perspective that the dialogical interactions generated are reflective of students' thinking, recent studies have shed light on students' understandings of important mathematical topics and practices (Koichu & R. Zazkis, 2013; D. Zazkis, 2014; R. Zazkis & D. Zazkis, 2016; D. Zazkis & Cook, *in press*). However, is this all that we can learn? The position taken in this paper is that the methodology affords not only a means to explore students' understandings but also to witness students' identities, if we attend to their perceptual ambiguity.

Figure 2. W. E. Hill's Cartoon

In 1915, W.E. Hill published the drawing shown in Figure 2. Staring at the drawing one of two images will appear, either a young lady with her head turned or an elderly woman looking down pensively. Both images are present. Yet, there is just one drawing. Both reside in the same set of lines. Yet, *we can only see one image at a time*. This is why the drawing has what psychologists call *perceptual ambiguity*. Perceptual ambiguity refers to instances in which one's grouping of certain contours, images, or ideas supports one's perception of a figure, object, or meaning while the grouping of other contours within the same image promotes a different singular perception. Such drawings were of interest to psychologists for they demonstrated that vision is an active rather than passive process; what is seen is constructed by the viewer actively. In this paper, perceptual ambiguity is of interest for it aptly describes an affordance of proof scripts: they afford observation of students' ways of attending to proofs while at the same time enabling us to witness students' identities, if we engage actively in the process of *seeing* students' positioning of themselves in relation to others, to the discipline, its practices and knowledge.

The Study

To examine students' ways of attending to contradictions, 43 proof scripts were collected from 2nd and 3rd year university students enrolled in an IBL - Introduction to Proof course who were given the proof task shown in Figure 3 during the last week of the term. Data collection occurred at a designated Hispanic-serving institution, where the majority of students are first generation college students, who qualify for need-based financial assistance. The original research question was "Which problematics and key ideas are salient to and noticed by students, when producing scripts for proofs involving contradictions?" However, when analyzing the data it became increasingly apparent that the discursive interactions did more than provide a window into students' reasoning about contradictions, for they also afforded an opportunity to witness students' enacted identities. In other words, the proof scripts embodied a form of perceptual ambiguity. Taking the position that students' identities are an important learning outcome of *all* mathematics courses, this affordance became the focus of the research. The purpose of this preliminary report is to provide an existence proof of proof scripts' perceptual ambiguity and in so doing establish a methodological approach to the study of students' identities that is distinct from but in harmony with the discursive approaches taken by Bishop (2012) and Setati (2005).

Assignment:

Part 1. Start by reading the proof and identifying what you believe are the "*problematic points*" for a learner when attempting to understand the theorem and its proof. A *problematic point* is anything you think is incorrect, is confusing, or is correct but warrants further discussion. List these "*problematic points*" in a bulleted list.

Part 2. Write a dialogue between you and Gamma in which you and Gamma discuss the theorem and its proof. The dialog should address the *problematic points* you identified (and listed in your bulleted list) through questions posed either by you or Gamma.

***Theorem*:** For any real numbers x and y, if $x \leq y$ and $y \leq x$ then $x = y$.

***Proof*:**

1) Assume x and y are real numbers such that $x \leq y$ and $y \leq x$.
2) Then ($x < y$ or $x = y$) and ($y < x$ or $y = x$).
3) We will consider four cases
   Case 1. $x < y$ and $y < x$.
   Case 2. $x < y$ and $y = x$.
   Case 3. $x = y$ and $y < x$.
   Case 4. $x = y$ and $y = x$.
4) In Cases 1 through 3 our assumptions contradict the Law of Trichotomy.
5) We are left with Case 4.
6) Case 4. $x = y$ and $y = x$.
7) Therefore, $x = y$.
8) The result follows. ♦

Figure 3. The Proof Script Task

The remainder of the paper, which will constitute the findings of the study, will proceed in two parts. In the first part, we examine excerpts from proof scripts to demonstrate how they afford the opportunity to examine students' understandings of the attended to problematics and content.

In the second, the same excerpts are analyzed to witness students' identities. Here it is important to note that the term *witness* is used intentionally and in reference to the belief that, at best, researchers can only hope for *disciplined subjectivity* (LeCompte et al., 1999) when seeking to understand others' identities and histories.

Findings, Part I: Proof scripts as a means to examine student understandings

While it may seem that there is not much to say in relation to the proof, the IBL-Introduction to Proof students had little difficulty elaborating on the given, arguably brief, argument. The majority of students produced discourses in which either Gamma or the student elaborated on why four cases were called for. Furthermore, many explained not only the generalized Law of Trichotomy but also the role of axioms and/or definitions in mathematics. To see this we consider Excerpt A, in which Student A explains the Law of Trichotomy to Gamma.

Excerpt A

Gamma:	*Why can't the first three cases be true?*
Student A:	*Because of the law of trichotomy.*
Gamma:	*What's the law of trichotomy?*
Student A:	*The Law of Trichotomy is an axiom we use. An axiom is a statement that is regarded as being the truth or accepted as true. So this axiom states that only one of the three following cases may happen: either $x < y$, $y < x$, or $x = y$. Applying this knowledge we can see why the first three cases don't work.*

This brief excerpt demonstrates several key understandings: (1) the status of the Law of Trichotomy within the theory of the real numbers; (2) the status of axioms within the discipline of mathematics; and, (3) a perhaps tentative understanding of contradictions – namely that they indicate an inconsistency has occurred within a mathematical theory, which must be resolved by deference to that which is taken to be true (i.e., given a choice between a result and an axiom we choose the axiom and label the result "*false*" or "*impossible*"). Beyond students' understanding of the components of mathematical theories, the proof scripts also often indicated students' understandings of a generalized proof structure. In particular, several scripts included the proofs for Cases 1 and 2, often citing basic axioms and definitions, and an explanation as to why Case 3 was not needed, if a proof of Case 2 was given.

Excerpt B

Student B:	*What's the word Mockingbird?*
Gamma:	*No much ese, just working on this pinche proof!*
Student B:	*Chale, that stuff ain't easy homes.*
Gamma:	*Que no, check it out and see if I got this mierda right?*
Student B:	*It looks good Homes except in linea 2 and 3 you got no detail ese. You need to explain cases 1 – 3!*
Gamma:	*Don't yell at me ese!*
Student B:	*Stop acting like a chavala!*
Gamma:	*Whatever homes!*
Student B:	*Anyways, lleva, with Line 2 you didn't state Axiom 6 which lets you split the inequalities foo.*
Gamma:	*Chingada madre! I forgot Axiom 10 in Line 3 for the cases 1, 2, 3.*
Student B:	*And for Line 4 you forgot Definition 6. Another thing for Case 2, if you prove it by contradiction using Axiom 10, because $x \neq y$, Case 3 is found without loss of generality because of Case 2.*

As is the case with Excerpt A, Excerpt B sheds light on several key understandings that are employed by the student. First, the remarks clearly indicate an understanding of the symbol $\leq$ and the fact that it has a mathematical definition which takes the form of a disjunctive statement; i.e., a form that justifies the partitioning of the proof into cases. Second, the script indicates the student has an observable understanding of a practice that is important to proofs at this level; namely, that sub-proofs which are identical in structure are not replicated within in a proof.

Findings, Part II: Proof scripts as a means to witness students' identities

When producing a dialog, a student must decide on how to position the interlocutors, their knowledge and status, goals and relationships. Moreover, the students must share or question specific actions taken within the proof and respond to questions about those actions by either drawing on interlocutors' knowledge and ways of reasoning or their interpretations of the expectations of participants of a discipline. Hence, by attending to students' positioning, use of language, characterizations of practice or articulation of expectations within a dialog, proof scripts afford the opportunity to perceive the inter-relationships between identity, knowledge, and practice experienced by students. It is for these reasons they afford opportunities to witness students' identities. Consider for example, Excerpt A. In the dialog, Student A is not positioned as an unknowledgeable or uncertain peer (a stance taken by authors in several proof scripts) but rather is positioned as a knowledgeable other, who can decidedly determine the status of statements ("*The Law of Trichotomy is an axiom")* while also acknowledging the relative status of "truth" in mathematics ("*regarded as being the truth or accepted as true"*). Likewise consider, Excerpt B, where the student skillfully pinpoints key gaps in the proof and the axioms and definitions necessary to elaborate on those gaps, while at the same time maintaining the dialect common to students in the area – essentially translating sophisticated mathematical ideas into an urban dialect. Here we see not only evidence of a student's content knowledge but also evidence of the *blending of identities*: identities common to the discipline of mathematics, where attention to detail and structure reigns supreme, and identities common to our urban youth, which are expressed through specific temporal and situated dialects that employ terms outside of formal English and Spanish (e.g., *ese* means "that" in Spanish but is slang for "man" or "dude" in parts of Mexico and the southwestern United States). Viewed in this way, the dialect presents an instance of true ownership, for the mathematics has bridged the great divide between the institutional home of the discipline, where the practices of mathematics are both recognized and valorized, and a community that is often structurally excluded from the discipline. Hence, the bridging represents an expression of identity, where the individual has taken ownership of the mathematics through its acculturation (as opposed to the student's).

Discussion & Concluding Remarks

Recognizing that, as argued by others (Bishop, 2012) identities are not only an important educational outcome but also critical to learning (Boaler, 2002), I have sought to demonstrate how proof scripts' perceptual ambiguity affords an opportunity to witness students' identities. Perceptual ambiguity in this context refers to an affordance of a static artifact, when that artifact can convey particular meanings through one's intentional focus on particular attributes, yet convey a distinct set of meanings should one's intentional focus shift to other, present but not yet attended to, attributes. The artifact – the student's proof script – being static does not change but rather our goal oriented activities do when actively "seeing" the artifact. As such, this work proposes that proof scripts can serve as a productive means for examining students' identities.

Questions

1. Is *perceptual ambiguity* the appropriate construct for describing the dualities of proof scripts?
2. What issues are there with describing researchers' inferences of identity as *witnessing*?

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