Building on Covariation: Making Explicit Four Types of "Multivariation"

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Covariation and covariational reasoning have become key themes in mathematics education research. In this theoretical paper, I build on the construct of covariation by considering cases where more than two variables relate to each other, in what can be called "multivariation." I share the results of a conceptual analysis that led to the identification of four distinct types of multivariation: independent, dependent, nested, and vector. I also describe a second conceptual analysis in which I took the mental actions of relationship, increase/decrease, and amount from the covariational reasoning framework, and imagined what analogous mental actions might be for each of these types of multivariation. These conceptual analyses are useful in order to scaffold future empirical work in creating a complete multivariational reasoning framework.

Key words: covariation, multivariation, reasoning, mental actions

The construct of covariation and the cognitive activities involved in reasoning about it have become important themes within mathematics education research (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Moore, Paoletti, & Musgrave, 2013; Moore, Stevens, Paoletti, & Hobson, 2016; Oehrtman, Carlson, & Thompson, 2008; Thompson, 1994). Yet, work on *co*-variational reasoning has essentially been limited to examining two variables changing in tandem with each other, perhaps with time as a mediator to that relationship (explicitly or implicitly). By contrast, as students continue to higher levels of science and mathematics courses, they encounter contexts in which there are more than two variables potentially changing in relation with one another. Note that I use the term "variable" in this paper to generally mean any potentially varying numeric value, including values of real-world quantities and mathematical function inputs and outputs. This theoretical paper is meant to build on the construct of covariation by explicitly considering cases where more than two variables (in addition to time) relate to and change with one another, in what can be termed "*multivariation*." In particular, I share the results of a conceptual analysis in which I identified four different types of multivariation, each with its own potential set of mental actions for reasoning about it. I also share the results of a second conceptual analysis examining what "*multivariational reasoning*" might possibly look like for each type, in terms of analogous mental actions corresponding to those already documented for two-variable covariational reasoning. The results of these conceptual analyses are meant to scaffold future empirical work, by helping to inform study design and data analysis, which can be used to establish a complete multivariational reasoning framework.

Covariation and Multivariation

Over the past couple decades several researchers have been contributing to a carefully developed sense of what "covariation" means (Carlson et al., 2002; Castillo-Garsow, 2012; Confrey & Smith, 1995; Johnson, 2012; Saldanha & Thompson, 1998). The central theme to this work is that covariation consists of imagining "two quantities [i.e., variables] changing together" (Castillo-Garsow, 2012, p. 55) in which "they are changing simultaneously and interdependently" (Johnson, 2012, p. 315). The specific term "quantity" often has additional meaning beyond being only a numeric value, and usually implies a measurable quality of an object (Thompson, 1994). However, covariation has also been applied to purely mathematical functions that are not necessarily contextualized as relationships between physical quantities (Oehrtman et al., 2008; Thompson & Silverman, 2008). In this paper, I consider covariation of variables both in terms of physical quantities and mathematical functions.

An important part of covariation, regardless of the variables involved, is the concept of time (Castillo-Garsow, 2012; Oehrtman et al., 2008; Thompson, 2011). Sometimes time can be explicitly present in the covariation as one of the two real-world quantities, such as distance and time. However, even for two non-time variables, or for a mathematical function, $y = f(x)$, if one applies "smooth" covariational reasoning (see Castillo-Garsow, 2012), time must necessarily be involved in imagining the change in progress. The necessity of time resonates with the assertion in Oehrtman et al. (2008) that, "The idea of covariation is fundamentally that of parametric functions" (p. 38). Thus, for my purposes, if a variable "A" is said to be varying (or covarying with another variable), it can be thought of as changing in time, A(t). However, this change does not have to happen linearly in "real" time, but can be conceptualized to move forward quickly or slowly, or to move in reverse, or to pause at a given instant.

Multivariation in the Current Literature and Conceptual Analyses

I began thinking about the construct of "multivariation" during a study involving the limits of complicated expressions (Jones, 2015) and another study involving multiple, line, and vector integrals (Jones & Naranjo, 2017). These ideas were further stoked when I encountered the "partial derivative machine" at a RUME conference, in which it is not always possible to hold certain variables "constant" in order to use basic covariation (see Roundy et al., 2015). I also began to see in mathematics and science textbooks how many instances there were in which multivariation could be involved. I wish to make clear that I am in no way claiming to be the inventor of the notion of multivariation, and that ideas surrounding multivariation have, in fact, been present in the mathematics education research literature, including studies on the graphs of multivariate functions (e.g., Dorko & Weber, 2014; Martinez-Planell & Trigueros-Gaisman, 2012; Weber & Thompson, 2014), on partial and directional derivatives (Bucy, Thompson, & Mountcastle, 2007; Martinez-Planell, Trigueros-Gaisman, & McGee, 2014, 2015), and on multiple integrals (McGee & Martinez-Planell, 2014). However, the main reason I believe this paper is needed is that while ideas pertaining to multivariation are present in the literature, multivariation as a construct in and of itself has essentially been implicit. Thus, there is still a need to explicitly discuss what *multivariation* and *multivariational reasoning* might consist of.

This theoretical report consists of the products of two conceptual analyses (see Thompson, 2008) meant to form the basis of future empirical work. The first conceptual analysis, presented in this section, focuses on what possible types of multivariation might exist. (The second analysis is described in the next section). To perform it, I looked through a large set of mathematics, science, and engineering functions and formulas, found mostly inside textbooks (e.g., Hibbeler, 2012; Serway & Jewett, 2008; Stewart, 2015), and considered how the variables in them could be conceptualized as changing with respect to one another. This conceptual analysis led to the identification of four distinct types of multivariation: *independent, dependent, nested,* and *vector*.

Four Types of Multivariation

Here I describe the four ways I identified that more than two non-time variables might be "changing together" in a potentially "simultaneous and interdependent" way (Castillo-Garsow, 2012; Johnson, 2012). I have stipulated non-time variables precisely because time is inherent in all types of variation, as discussed previously, whether univariation, covariation, or multivariation. Thus, time-parametric equations are not considered a separate type of multivariation, since they are already inherent in all types.

Independent multivariation. The first type of multivariation I describe, *independent multivariation*, is probably the most commonly imagined type of multivariation in mathematics because of how we often work with multivariate functions, like $z = f(x,y)$. In this type, there are multiple "input" variables (e.g., x and y) that each individually covary with an "output" variable (e.g., z), but where the "input" variables need not covary with each other. In other words, the covariations between each input variable with the output variable can be conceptualized as *independent* from each other. In contrast to covariation, a change in the output does not necessarily imply a change in one particular input, since the change in output could have happened as a result of covariation with a separate input variable. Next, I note that what counts as "input" and "output" does not necessarily need to be fixed (e.g., solving to get $x = f(y,z)$), so long as the covariations between each of the input variables and the output variable remain independent. I also note that this type of multivariation could be extended to include as many input variables as desired, such as for the function $z = f(x_1,x_2,\ldots,x_n)$.

Since each input variable covaries with the output variable, it might be tempting to think of this type of multivariation as simply basic covariation by holding all but one of the input variables constant at a time. While that can be true, what makes this distinct from two-variable covariation is that it is, in fact, possible to imagine *all* of the input variables changing at the same time, each having their own impact on the output variable. This is similar to the idea of directional derivatives (see Martinez-Planell et al., 2015), or to taking a surface defined by $z = f(x,y)$ and tracing out a curve on it by parameterizing $x(t)$ and $y(t)$ over the interval $a \leq t \leq b$.

This type of multivariation is present in many science contexts involving real-world quantities. The key is whether it is realistically and conceptually reasonable to hold certain variables constant while varying others. For example, force (an output variable) can be defined as the product of mass and acceleration (the input variables), as in $F = ma$. In this case, one can imagine holding m constant and changing a to produce changes in F, or holding a constant and changing m. Yet, what makes this "multivariation" rather than "covariation" is that m and a could be imagined to be changing simultaneously, yet independently, each producing concurrent changes in F. Note that m and a do not have to be the input variables, since one could imagine holding F constant and changing m to produce changes in a.

Dependent multivariation. The second type of multivariation, *dependent multivariation*, more commonly arises in real-world contexts, since input variables for mathematical functions are typically conceptualized as, literally, "independent variables." However, for certain scientific contexts it might not make sense to conceive of holding some variables constant while the others vary. In fact, some science educators have already brought up this idea, since "holding constant" is not always possible (e.g., see Bucy et al., 2007; Roundy et al., 2015). The main idea for this type of multivariation is that, rather than having several independent covariations between multiple "input" variables and a single "output" variable, a change in any variable produces simultaneous changes in *all* other variables. Further, as those other variables change, they also immediately induce changes in all other variables in the system.

For example, if a fixed amount of gas is contained in a flexible balloon, the ideal gas law models the relationship between the volume, V, pressure, P, and temperature, T, of the gas through the equation $PV= kT$. However, unless certain laboratory conditions are imposed, it might not be realistic to hold P constant while T and V change with respect to each other. More realistically, if the temperature increases, the pressure and volume both increase simultaneously and their changes can feed back into the system immediately. Or, to pull from a rather different context, suppose an economist is examining how price, affected by demand and supply, is changing for a particular good in a market that is in flux. Again, it might not be realistic to imagine holding demand constant in order to manipulate supply and measure the corresponding changes in price. As the supply changes, both price and demand may change simultaneously as the market approaches a new equilibrium.

To be clear, in this type of multivariation, I am not saying that it is not *mathematically* possible to hold one of the variables constant in order to enact calculations. However, my point is that these types of contexts cannot conceptually be fully accounted for *only* through multiple independent two-variable covariations. Rather, one would have to use mental actions that involve multiple variables all having simultaneous impacts on each other in order to reason accurately about the real-world processes.

Nested multivariation. The third type of multivariation I describe, *nested multivariation*, comes from how one might conceptualize changes when the relationships between variables are based on the structure of function composition, such as $z(y(x))$ (for more on student understanding of function composition, see Ayers, Davis, Dubinsky, & Lewin, 1988; Breidenbach, Dubinsky, Hawks, & Nichols, 1992). For $z(y(x))$, if one imagines changes in x, then there are corresponding changes to y. Yet those changes in y now correspond to changes in z. While it is true that one can, in fact, think of direct two-variable covariation between x and z, nested multivariation conceptualizes the relationship as having intermediary variables. Thus, the difference between whether it is two-variable covariation or nested-variable multivariation is *not* inherently dependent on the structure of the formula or function. Rather, it is necessarily a product of how one *conceptualizes* the changes taking place. For example, for the equation $y = \sin^2(x)$, it is true that one can imagine x and y changing directly with each other. However, it is also possible to imagine that as x increases, from say 0 to $\pi/2$, it produces corresponding increases in the value of "$\sin(x)$," and that as the value of $\sin(x)$ increases, it in turn generates increases to the "$\sin^2(x)$" values. In other words, as one variable changes it induces a change in a second, which induces a change in a third variable (and potentially so on to include as many variables as desired).

To describe an example from science, consider the formula from relativity relating velocity, v, with the relative mass of an object, m, given by $m = m_o\big/\sqrt{1-(v/c)^2}$ (c is the speed of light and m_o is the relative resting mass). When I have asked students to describe what happens to mass as v approaches c, they tended to think through this formula piece by piece. They would first discuss how an increasing v made the ratio between v and c approach one. They would then discuss how that corresponded to $\sqrt{1-(v/c)^2}$ shrinking to zero, which lastly made the value of the entire expression tend toward infinity. To represent their thinking in mathematical notation, they essentially thought of the mass equation broken down into a ratio function, $\beta(v) = v/c$, which became an input for the Lorenz factor, $\gamma(\beta) = 1\big/\sqrt{1-\beta^2}$, which in turn became the input for the mass, $m(\gamma) = m_o\gamma$. As explained previously, it is true that one can think of direct covariation between v and m. If one does so, then in that case they are employing covariational reasoning. However, if they imagine nested changes from v to β to γ and finally to m, then I argue they are employing nested multivariation reasoning.

Vector multivariation. The last type of multivariation I describe, *vector multivariation*, may be the most cognitively complex and gets its name because it deals with multiple independent inputs each simultaneously associated with multiple independent outputs (i.e. a vector function).

Thus, vector multivariation is essentially a generalized version of independent multivariation in that it consists of several *independent multivariations* each happening *independently* of each other. For a vector function, $\vec{F}(x, y) = \langle u(x, y),\ v(x, y) \rangle$, like with independent multivariation, one can think of holding, say, y constant and letting x vary, but in this case that variation corresponds to changes in both u and v at the same time. Further, imagining both x *and* y varying simultaneously leads to four pairs of independent covariations that could potentially need to be cognitively managed all together. As with all other types of multivariation, vector multivariation could be extended to include as many variables as desired, including several input or several output variables.

For examples of vector multivariation, consider a vector field mapping $\mathbf{R}^2$ to $\mathbf{R}^2$. If one were to take a starting point (x,y) and increase the x-coordinate, tracing a horizontal line through the vector field, both the horizontal and vertical components of the vector field could be changing simultaneously. Similarly, if one increased the y-coordinate and traced vertically through the vector field, both components of the vector field could change. Now, for full vector multivariation, if one traced out a curve, C, in the x-y plane along which both x and y are changing simultaneously, one would have to coordinate how much x and y are each changing, and what the resulting changes in the horizontal and vertical components of the vectors are. This type of multivariation shows up in vector integrals, $\int_C \vec{V} \cdot d\vec{r}$, and also occurs for functions with complex inputs and outputs, $f: C \to C$. For complex functions, if the input complex variable changes along a curve in the complex plane, one would have to simultaneously attend to changes in the real and imaginary parts of the input variable, as well as the changes in the real and imaginary parts of the output variable. In science, this type of multivariation could be present in any context involving vector spaces, such as gravitational fields or electrical fields. One could imagine a particle tracing some path through those fields, with changes happening in each of the vector components as the path is traced out.

Comparison of structures. To summarize this first conceptual analysis, Figure 1 shows the distinct conceptual structures for the four different types of multivariation (and covariation). Of course, each type of multivariation could be extended to include as many variables as desired.

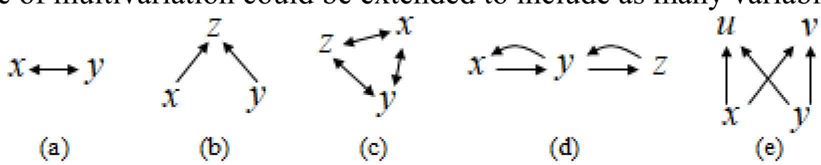

Figure 1. Comparison of structures for (a) basic covariation, (b) independent multivariation, (c) dependent multivariation, (d) nested multivariation, and (e) vector multivariation, where (b)–(d) could each be extended to include as many variables as desired.

Covariational Reasoning and Multivariational Reasoning

To describe the second conceptual analysis, I briefly return to basic two-variable covariation. Carlson et al. (2002) described five mental actions that pertain to increasingly sophisticated levels of covariational reasoning. The first three mental action levels are given as (p. 357): (1) "Coordinating the value of one variable with changes in another," (2) "Coordinating the direction of change [i.e., increase or decrease] of one variable with changes in the other variable," and (3) "Coordinating the amount of change of one variable with changes in the other variable." For my purposes, I label these three mental actions as "relationship," "increase/decrease," and "amount." The fourth and fifth mental action levels then progress to changing rates of change, marking a shift from reasoning about the two variables directly to reasoning about how a rate of change itself varies. For my conceptual analysis, I focused on what mental actions for each type of multivariation might be analogous to the *relationship*, *increase/decrease*, and *amount* mental actions from covariation. I do not include changing rates of change in this conceptual analysis at this point because of the potential complexity of multiple simultaneous changing rates of change. Rather, my conceptual analysis focuses on providing an initial step into how one might imagine the variables themselves in the system and their direct relationships with each other.

Analogous Multivariational Reasoning Mental Actions

Here I describe the potential mental actions of multivariational reasoning that might be analogous to relationship, increase/decrease, and amount from covariation. This "thought experiment" is intended to scaffold possible empirical methods aimed at examining the nature of multivariational reasoning, by imagining beforehand what cognitive activities might specifically be targeted in empirical research.

First, what might be the mental actions in *independent multivariational reasoning* analogous to relationship, increase/decrease, and amount? The first mental action would likely consist of a realization that multiple input variables may impact a single output variable, and that each change may be happening in isolation or simultaneously. In thinking of the surface defined by the graph of $z = f(x,y)$, it would be the realization that one can trace a path along this surface in *any* direction, freely. The next mental action may consist of coordinating each individual change in the input variables to an overall net *directional change* for the *inputs*. In the case of $z = f(x,y)$, this would be congruent to imagining a "change vector," ΔV, whose components are $<\Delta x,\Delta y>$, though I use the word "vector" for convenience and note that a student would likely not conceptualize it as an actual "vector." In contrast to the second level of covariational reasoning, where there is already attention to whether the output increases or decreases, I hypothesize that this is a required preliminary mental action for independent multivariational reasoning, not yet involving "increase/decrease." That is, it may be required to simply identify the direction of the change vector *before* determining whether the output increases or decreases along it. It would then be a separate mental action in which one would coordinate this *change vector* with whether the output variable increases or decreases. Thus, we can see additional sophistication in independent multivariation reasoning above what is required for two-variable covariational reasoning. Only after these three mental actions would a fourth mental action coordinate the *amount* of change in the output variable along the direction of this change vector.

Next, consider *dependent multivariational reasoning*. Here, the first mental action would likely consist of the coordination of a change in one variable with simultaneous and interdependent changes in all other variables in the system. That is, it would be the realization that some variables cannot be held constant in a realistic way and that a system may only be understood by imagining all variables changing interdependently. The second mental action might then consist of coordinating the change in one variable with whether each of the other variables increases and/or decreases. This mental action is quite sophisticated, since one must coordinate interdependent increases and decreases, meaning it may even consist of separate mental actions. For example, in the balloon context, increasing temperature would mean an increase in pressure, but the fact that the volume also increases means that the pressure would not increase by as much as would be predicted if volume were able to be held constant. The next mental action would consist of coordinating the change in one variable with the *amount* by which each of the variables in the system interdependently change as a result.

For *nested multivariational reasoning*, each mental action essentially deals with *chained* reasoning. The first mental action would involve coordinating a chain of changes from one variable to the next. It would be the realization that a change in one variable would have effects on a *sequence* of other variables. The second mental action may consist of coordinating the change in the first variable with whether the second variable increases or decreases, and coordinating whether increases or decreases in the second correspond with increases or decreases in the third, and so on. A possible metaphor is a sequence of gears where one attends to how a rotation in the first induces rotations on the others. Again, this may actually represent several separate mental actions. The next mental action would follow this same chain, but would coordinate *how much* each variable in the sequence increases or decreases.

For *vector multivariational reasoning*, the first mental action may consist of coordinating changes among several input variables with changes among several output variables. It would be the realization that multiple input variables can impact multiple output variables, in isolation or simultaneously. The second mental action, like with independent multivariation, would likely not deal with whether the output variables are increasing or decreasing, but would consist of a preliminary mental action of coordinating the changes in the input variables to form an overall "input change vector," ΔV_{in}. This input change vector defines the direction along which the change is happening. I believe that it may then require several mental actions to achieve complete analogs to the increase/decrease and amount mental actions. The first of these would be to coordinate whether each output variable increases or decreases in the direction of the input change vector. The second would be to coordinate the amount of change in each of the output variables. The third would be to coordinate the changes in each of the output variables to create an overall "output change vector," ΔV_{out}. These may then culminate into a fourth mental action that directly coordinates the *input* change vector, ΔV_{in}, with the *output* change vector, ΔV_{out}.

Lastly, I note that for independent and vector multivariation, it is possible to consider the direction of change first, such as imagining tracing along a curve, C (like in line and vector integrals). In this case, some of the mental actions may reverse, and rather than construct the input change vector from changes in the inputs, the mental actions might consist of *decomposing* the change vector into changes in the inputs.

Conclusion

In this paper I described conceptual analyses into different types of multivariation. I also described mental actions potentially associated with each type of multivariational reasoning and how they might be different from each other and from two-variable covariational reasoning. The usefulness of this report is in producing a conceptualization of multivariation that can provide the basis and framing for empirical studies into the nature of multivariational reasoning, such as ensuring that each hypothesized mental action is targeted during the study. I claim that the different types of multivariation described here are far more than theoretical curiosities. Students encounter, both in mathematics and in science, many contexts in which one of these types of multivariational reasoning might be needed. In fact, any context that involves more than two variables, which can even show up in pre-collegiate mathematics, may inherently require at least some of the more basic multivariational mental actions. As such, I believe this paper to be a useful step in understanding how reasoning about these contexts may be developed.

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