Proof Norms in Introduction to Proof Textbooks

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We present a textual analysis of three of the most common introduction to proof (ITP) texts in an effort to explore proof norms as undergraduates are indoctrinated in mathematical practices. We focus on three areas that are emphasized in proof literature: warranting, proof frameworks, and informal instantiations. Each of these constructs have been connected to students' ability to construct, comprehend, or validate proofs. We carefully coded all the proofs and supplemental material across common sections in the textbooks. We found that the treatment of proof frameworks was inconsistent. We further found that textbook proofs rarely used explicit warranting and informal instantiations. We conclude by reflecting on the impact of inconsistent proof norms and unsubstantial focus on supportive proof components for students in ITP courses.

Keywords: Proof, Proof Norms, Textbook Analysis, Warranting, Proof Frameworks

Proof is an essential aspect of the mathematical discipline (de Villiers, 1990; Hanna, 2000; Hersh, 2009; Rav, 1999), and as such proficiency in all areas of proof is important for students in undergraduate mathematics programs to gain. In order to meet this goal, university mathematics departments offer introduction to proof (ITP) courses to help students learn about the argumentative process of proof specific to mathematics. One of the main objectives of the ITP course is to improve the undergraduate student's ability to construct formal proofs. Despite this objective, numerous studies have documented the difficulties that students have in making the transition to advanced mathematics and in their ability to construct formal proofs (e.g. Moore, 1994; Selden & Selden, 2003; Weber & Alcock, 2004). In spite of this research and the hypothetical function of the ITP course, until recently little focus has been put on the nature of these classes.

In this study, we focus on the intended curriculum of ITP courses as reflected in textbooks. Curricular materials provide a significant factor in the learning and development of reasoning and proof (Konior, 1993; Stylianides, 2014). As such, the aim of this study is to understand both the implicit and explicit messages that ITP textbooks send to the reader about the nature of mathematical proof. Specifically, we analyzed three research-based aspects of proof: proof frameworks (Selden & Selden, 1995, 2003; Weber, 2009), explicit warranting (Alcock & Weber, 2005; Inglis & Mejiá-Ramos, 2008; Pedmonte, 2007; Toulmin, 2003), and diagrams as well as other informal reasoning (Samkoff, Lai & Weber, 2012; Weber & Alcock, 2004). We found that ITP proofs often lacked consistency in terms of frameworks and warranting, and generally overlooked diagrammatic and other informal reasoning.

Theoretical Framing and Background

Underlying our work is the assumption that curricular materials reflect and impact the nature of a mathematics course. In general, we argue in alignment with Zhu and Fan (2006):

> ...textbooks are a key component of the intended curriculum, they also, to a certain degree, reflect the educational philosophy and pedagogical values of the textbook developers and the decision makers of textbook selection, and have substantial influence on teachers' teaching

and students' learning (p. 610).

While not a perfect substitute for the classroom, textbooks provide a substantial set of example proofs that students experience. Furthermore, writers of these texts are implicitly endorsing a set of norms for proofs in this setting. In this way, textbooks provide an artifact for exploring the norms for ITP-level proofs and a substantial resource impacting students' ultimate learning.

We also frame our work in terms of proof as a social construct. Research has established that what constitutes a valid proof varies based on context and even from individual to individual (e.g. Moore, 2016; Weber, 2008). In this way, we may expect that textbooks may reflect similar variation in proof norms. We focus on three areas of norms: proof frameworks, warranting, and diagrams and other informal reasoning.

Proof Frameworks

Selden and Selden (1995) introduced the construct of *proof framework* to capture the "top-level" structure of proof that comes directly from unpacking a statement. For example, if the mathematical statement to be proven is, "For all integers x, if x is odd, then $x + 1$ is even integer" the *complete* proof framework for a direct proof of this this statement might look like the following:

Proof: Let x be an integer. Suppose x is odd…
…then $x + 1$ is an even integer. ■

Notice this is a direct proof. A contrapositive proof framework would unpack the contrapositive statement.

The literature reflects that ability to produce an accurate proof framework has a relationship to other activities such as constructing, validating, and comprehending proof (e.g. Mejiá-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012; Selden & Selden, 1995, 2003; Weber, 2009). Both Selden and Selden (1995) and Weber (2009) found that many students do not typically check proof frameworks and may lack awareness as the essential role of an appropriate proof framework in a proofs' validity.

Warranting

One of the most fundamental acts in mathematics, especially in proof and proving is that of warranting: justifying assertions (see Hanna 1991, 1995; Healy & Hoyles 2000). According to Toulmin (2003) arguments, and by extension proofs (see Alcock & Weber, 2005; Inglis & Mejiá-Ramos, 2008; Pedmonte, 2007), have a formal structure which is defined by the interplay of at least three fundamental constructs; that of *data*, *warrants*, and *claims*. In the tradition of Toulmin (2003), a claim is a statement or assertion of what is true, the data is the grounds by which the assertion of truth is made, and the warrant justifies the connection between the data and the claim by, for example, invoking a definition or rule. For instance, we can use Toulmin's scheme to analyze the following statement: "Since x is odd, then by the definition of odd, $x = 2n + 1$ for some $n \in \mathbb{Z}$." The claim being made in this instance is that "$x = 2n + 1$ for some $n \in \mathbb{Z}$," the data on which rests the truth of this assertion is "Since x is odd," and the warrant that connects the data and claims is "by the definition of odd."

Warranting, explicitly connecting data and claim, is an important ability for students to learn. Alcock and Weber (2005) asserted that, "Failure to consider the warrants used in a proof will not only cause students to be unable to validate proofs reliably, but… can also prevent them from gaining conviction and understanding from proofs presented in their classrooms" (p. 133). Furthermore, Alcock and Weber claimed that instructors for proof-oriented course do not commonly discuss warrants and that textbooks are also infrequent in explicit language on the subject. Within proofs, warrants are often left implicit. The ability to infer these implicit warrants is an essential skill for understanding proofs in advanced mathematics and should be part of students' enculturation into proof based mathematics (Weber & Alcock, 2005).

Diagrams and Other Informal Reasoning

Informal reasoning plays an important role in the learning and construction of proofs (Fischbein, 1983; Hanna, 1991), moreover, it is the multifarious interplay of these intuitions with the rigorous and abstract aspects of mathematical ideas that are the cornerstone of advanced mathematics (see Mariott, 2006). Informal reasoning may take many forms including that of exploring examples or diagrams.

Diagrams and other example based instantiations can aid students in understanding a statement, and gaining a level of conviction in a theorem and its proof (see Alcock & Weber, 2008, 2010; Samkoff, Lai, & Weber, 2012, Weber & Mejiá-Ramos, 2015). As moving from informal to formal reasoning is an important factor in the creation of mathematical ideas (Raman 2003; Weber & Alcock, 2004), we explored the degree to which textbooks leveraged informal instantiations including: using numbers to explore computational cases (e.g., substituting values as test cases), building example sets to explore set interactions (e.g. unions, intersection, Cartesian products), or testing the behavior of specific set members under a particular mapping.

Methods

Textbook Sample

In this study we analyzed three textbooks (see Table 1) which are among the most used textbooks for the standard ITP course in the United States. According to David and Zazkis (2017), these textbooks represent roughly 27%[1] of the market share for textbooks used by departments and instructors for the ITP course. All other standard ITP texts had less than a 4.2% market share.

Table 1. Introduction to Proof Textbooks

Title	Publisher	Authors	Market Share	Year
Mathematical Proofs: A Transition to Advanced Mathematics (**Book A**)	Pearson Education	Chartrand, Polimeni, & Zhang	11.3%	2013
A Transition to Advanced Mathematics (**Book B**)	Brooks/Cole	Smith, Eggen, & St. Andre	10.6%	2011
Book of Proof (**Book C**)	Richard Hammack	Hammack	4.9%	2013

We selected the sections and chapters that aligned with content found in a typical Standard ITP course: formal logic, number and set theory, relations, functions, and cardinality of sets (David & Zazkis, 2017). We grouped the sections into *introductory material* which consisted of all sections prior to the three *content specific* sections of functions, relations, and cardinality of sets. For each of the pertinent sections and or chapters of the textbooks, we read the vast majority of proofs, as well as any explicit commentary that each book provided about the construction of said proofs. All told, we analyzed arguments for 345 mathematical statements, of which we identified 280 were proofs or at very least the outline of a proof. See table 2 for a breakdown of number of proofs by section in each text.

[1] David and Zazkis (2017) shared information on 154 universities use of textbooks, 12 of which used lecture notes only, meaning that 38 of the remaining 142 classes used one of these three texts.

Table 2. Mathematical Statements and Proofs

	Book A		Book B		Book C		Total	
Section	Statement	Proof	Statement	Proof	Statement	Proof	Statement	Proof
Intro. Material	56	50	85	64	46	46	187	160
Relations	9	9	17	15	3	3	29	27
Functions	10	10	34	23	6	4	50	37
Cardinality of Sets	21	17	44	30	14	9	79	56
Overall	96	86	180	132	69	62	**345**	**280**

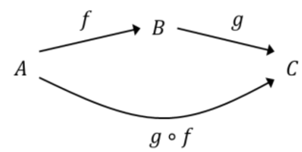
Each of the three textbooks were analyzed using thematic analysis (Braun & Clarke, 2006). The analysis began with open coding proofs from Book A within the categories of proof frameworks, warranting, and diagrams/informal reasoning. This set of codes of was condensed and categorized. The robustness was tested in the next text: Book B. These coding scheme was further expanded when new codes emerged from this text. For the purpose of consistency and in being faithful with the method of constant comparison, prior to coding Book C, we recoded Book A using the full set of codes. Once the second coding of Book A was completed and codes were refined and condensed, coding of Book C began, and it was at this point that saturation occurred and the coding cycle ceased as no new codes arose from coding Book C.

Analytic Framework

In this section we share the most relevant sections of our coding framework. Proof frameworks were coded as either complete, incomplete, or non-existent. All explicit warrants were identified based on their type: definition, theorem, or algebra. We identified two types of informal categories: diagrams (visual representations) and informal reasoning. See Table 3 for elaborations of the codes,

Table 3. Analytic Framework for Coding Proofs

Category	Code: Description	Example
Proof Frameworks	**Complete:** A proof has both the antecedent and consequent of the original statement represented in accordance with the proof method being employed **Incomplete:** A proof which only has one or the other of the antecedent or consequent represented in accordance with the proof method being employed. **Non-Existent:** A proof which has neither the antecedent nor consequent represented.	For all integers x, if x is odd, then x + 1 is even integer. *Proof:* Let $x = 2n + 1$ for some $n \in \mathbb{Z}$… …then $x + 1$ is an even integer. ■ Code: Incomplete (the antecedent is not explicitly unpacked to: "Let *x* be an integer."
Warrants	**Definition**: The authors use a definition, property, axiom, or other fact *accepted* in the text as a warrant to connect some data and claim within a proof.	"By the distributive property we have that $3 \cdot (x + y) = (3 \cdot x) + (3 \cdot y)$…"

	Theorem: The authors use a theorem, corollary, lemma, or other fact *proven* in the text as a warrant to connect some data and claim within a proof.	Code: Algebra (reference to the field property distributive)
	Algebra: The authors use an algebraic field axiom as a warrant to connect some data and claim within a proof.	
Informal	**Strategies**: Any instance where the authors present syntactic strategies for proof production.	
	Semantic: Any instance where a proof or its supplementary material references a diagram or present other semantic explorations whether to simply clarify an idea or as a means to further the proof.	Code: Diagram (this figure was directly reference in a proof).

Results

Proof Frameworks

We found that the three books varied in terms of how often they presented a complete proof framework (CFP). Book A and Book B provided CFP roughly 1/3 of the time while Book C provided CFP 68% of time (see table 4). Roughly a quarter of proofs from Book A and Book B did not include either the proof framework antecedent or conclusion. A student reading Book C is exposed to significantly more complete proof frameworks than a student reading Book A or Book B.

Table 4. Complete Proof Frameworks and Non-Existent Proof Frameworks

Text	Complete	% Complete	Non-Existent	% Non-Existent
Book A	32	37%	22	26%
Book B	50	38%	27	20%
Book C	42	68%	2	3%
Overall	124	44%	51	18%

Additionally, we found that complete frameworks were present more consistently in the introductory materials, as proof methods were being explicated, and then used less and less as we continued through each book (see Table 5). Conversely, non-existent frameworks were less frequent in the introductory material, but become more common as the texts progress. Finally, proof frameworks in the supplementary material were treated in a manner roughly parallel to how they were treated in the body of the proofs themselves.

Table 5. Complete Proof Frameworks (Comp.) by Topic

	Book A		Book B		Book C	
Sections	Comp.	% Comp.	Comp.	% Comp.	Comp.	% Comp.
Intro. Material	25	50%	34	53%	33	72%
Relations	2	22%	2	13%	3	100%

Functions	3	33%	2	9%	3	75%
Cardinality	2	12%	12	40%	3	33%

Warranting

We found explicit warranting to be an uncommon occurrence in all three textbooks. In Table 6, we present the use of explicit warrants within the categories of definitions (DEF), theorems (THM), algebraic field axioms (ALG). We also provide the number of proofs and number of statements. Overall only 6% of statements included explicit warranting throughout all three texts, and a little more than a third of all proofs had *any* explicit warranting of any kind in them. Thus the reader of any of these texts is unlikely to regularly be exposed to explicit warrants. In all three texts, field axioms were implicitly warranted in all cases. Book C provided the most warrants often explicitly warranting with definitions.

Table 6. Proofs with Warrants, Statements, and Total Warrants in ITP Texts

	Proofs with Warrants	Statements in Proofs	Warrants				Total by Statement (%)
			DEF	THM	ALG	Total	
Book A	18	808	5	19	0	24	3%
Book B	47	885	21	38	0	59	7%
Book C	37	633	51	20	0	71	11%
Total	102	2326	77	77	0	154	6%

When we expanded our analysis to the supplemental material, we found that Book B and Book C often included warrants in parenthetical comments. An additional 46[2] explicit warrants can be found in parenthetical comments. This reflects that warranting is (a) not meant to be part of the proof product; (b) but warranting is part of the proving process.

Informal Reasoning and Diagrams

We found diagrams and other semantic explorations to be the most sparsely represented construct of all coded entities. Conversely, the authors regularly offer insight akin to Weber's (2001) strategic knowledge as strategies were especially prevalent. In Table 7 we present the use of strategies and semantic explorations as part of the argumentative process. The authors show a bias toward presenting strategies for how to produce a proof rather than the exploring instantiations to better understand the underlying premise being proven.

Table 7. Use of Informal Reasoning and Diagrams

	Proofs	Strategies	Strategies (%)	Semantic	Semantic (%)
Book A	86	40	47%	6	5%
Book B	132	72	55%	8	6%
Book C	62	10	16%	3	5%

[2] Three of these warrants were related to algebraic field axioms.

Total	280	122	44%	15	5%

Discussion

Through our textbook analysis, we found proof frameworks, warranting, and informal reasoning occurred inconsistently and often infrequently in typical ITP textbooks. Proof frameworks were by far the most treated entity of the three as each text explicitly touched on the idea that there is an overarching logical shell implied by the mathematical statement to be proved and the chosen proof method. This was a point that was touched on early by each of the three texts, but was not used consistently throughout the texts. In general, the texts convey a message that a proof framework need not be explicit part of a proof. As we know students struggle to produce, identify, and understand the role of proof frameworks (Mejía-Ramos et al., 2012; Selden & Selden, 1995, 2003), leaving such a framework implicit may be further increasing the difficulty in seeing the importance of these structures.

Our analysis bore out Alcock and Weber's (2005) conjecture that textbooks do not treat warranting and its importance explicitly. None of the three texts explicitly addressed the role of warranting. Further, explicit warrants were a rarity. The textbooks reflected a norm for the ITP setting that field axiom claims never need warrants. Claims relying on definitions or theorems sometimes need explicit warrants. There is no reliable message to be found concerning the use of warranting within these texts. For the reader this means that coming to a deeper understanding of explicit warrants will be difficult, let alone coming to have an understanding about the importance of being able to infer warrants. This means that the impetus is on the instructors of ITP courses to introduce what a warrant is and the role that it plays in the argumentative process, but also to explore how they are used implicitly and the role the reader of a proof has in inferring the implied warrant (see Alcock & Weber, 2005, Weber, 2004).

Finally, informal reasoning, much like that of warranting, had no explicit conversation surrounding the subject in any of the three texts. None of the three texts present a clear picture of how informal reasoning may guide the production of formal proof and the place that informal ideas and representations such as diagrams play in a formal proof. The exception to this occurs in the cardinality section where maps are written informally. This more informal treatment is leaves us to wonder the lasting affect that seeing a large body of formal proofs, followed by some very informal proof on a particular subject will have on students' ability to understand and gain conviction in ideas surrounding cardinality in their future classes.

Our study implications are limited in the degree to which commonly used textbooks reflect the implemented curriculum. We see this exploration as highlighting that there is a clear under treatment of warrants and information exploration in common textbooks. We do not wish to claim this means there is an under treatment of these topics within classes. However, our textbook analysis paired with research on the typical nature of proof-based courses (e.g. Alcock & Weber, 2010; Lew, Fukawa-Connelly, Mejía-Ramos, & Weber, 2016), builds a strong argument for this being the case.

Furthermore, the textbook analysis unearthed inconsistent proof norms particularly around proof frameworks and warranting. If a set of typical textbooks designed to enculturate students in proof production contain fundamental inconsistencies, how can we expect our students to understand the importance of these constructs? As instructors and researchers, we must be aware of the messages our curricular materials may send.

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