Transforming students' definitions of function using a vending machine applet

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The purpose of this study is to examine the understandings of functions that students developed and tested while engaging with a Vending Machine applet. The applet was designed to purposefully problematize common misconceptions associated with the algebraic nature of typical function machines. Findings indicate that the applet disrupts students' algebraic view of function and supports their transformation of meaning schemes for the function concept.

Keywords: Functions, Calculus, Teaching with Technology

The concept of function is central to the study of undergraduate mathematics, science, and engineering (e.g., Cooney, Beckmann, & Lloyd, 2010; Dubinsky & Harel, 1992; Leinhardt, Zaslavsky, & Stein, 1990). However, research has revealed persistent and common misconceptions among undergraduate students with respect to the definition of function (Vinner & Dreyfus, 1989), use of function notation (e.g., Oehrtman, Carlson, & Thompson, 2008), and connections between function representations (e.g., Brenner et al., 1997; Clement, 2001; Dreher & Kuntze, 2015; Stylianou, 2011). Hence, there is a need for the development and study of interventions to help address misconceptions such as these among undergraduate students so that they are set up for success in their future studies.

To this end, we designed and studied the implementation of an applet-based learning intervention focused on disrupting undergraduate students' understanding of the function concept. The purpose of this study is to examine the understandings of functions that students developed while engaging with a Vending Machine applet designed for students to test and develop their own definitions for function.

Background Literature

Much of the research on student understanding of function has occurred in the context of college algebra, precalculus, or calculus classes. Through these studies there has been a careful identification of common understandings that students develop related to the concept of function. One common student understanding is that functions are defined by an algebraic formula (Breidenbach et al., 1992; Carlson, 1998; Clement, 2001; Sierpinska, 1992). This is not surprising since functions are typically introduced as specific function types, such as linear and quadratic functions, in the middle school and high school curriculum (Cooney et al., 2010). Thompson (1994b) found that not only do students view functions as algebraic formulas, they often view functions as two expressions separated by an equal sign. While an equation view of function is not inherently wrong, it is narrow and can lead to difficulties for students as they work with functions in different contexts and with different representations (Cooney et al., 2010).

Along with an algebraic view of functions as representations of particular objects (e.g., graphs, expressions) rather than a relationship between inputs and outputs, research has also shown that students often rely on the graph of an equation and the vertical line test to differentiate a function from a non-function (Breidenbach et al., 1992; Fernandez, 2005). This can lead to conceptual difficulties in determining functions from non-functions, including the tendency to apply procedures to determining functions from non-functions (Breidenbach et al., 1992; Fernandez, 2005). Students whose view of function is algebraic and who use procedural techniques to identify functions and non-functions struggle to comprehend a general mapping of input values to a set of output values (Carlson, 1998; Thompson,1994a). The consistency of problematic understandings of function found across studies of students speaks to the need for pedagogical practices to specifically disrupt and correct these ideas. This is especially important given that function is a unifying concept among many undergraduate mathematics courses. Students need to understand both what a function is (i.e., the definition of function) and how to identify one across contexts and representations (e.g., Carlson, 1998; Thompson, 1994a; Breidenbach et al.,1992). Yet, particular attention to students' understanding of the definition itself has not been widely researched.

Theoretical Framework

As we consider undergraduate students' learning related to function, we adopt a theoretical lens of transformation theory (Mezirow, 2009). Transformation theory is consistent with constructivist assumptions, specifically that meaning resides within each person and is constructed through experiences (Confrey, 1990). Mezirow (2009) describes four forms of learning that lie at the heart of the theory: elaborating existing meaning schemes, learning new meaning schemes, transforming meaning schemes, and transforming meaning perspectives (p. 22). Meaning schemes are the specific expectations, knowledge, beliefs, attitudes or feelings that are used to interpret experiences (Cranton, 2006; Peters, 2014). In the context of this study, an undergraduate student might transform his/her meaning scheme for function by rejecting her prior conception of function as a graph that passes the vertical line test and adopt a broader view of function that includes numerical and algebraic representations.

Learning by transforming meaning schemes often begins with a *disorienting dilemma*. This stimulus requires one to question current understandings that have been formed from previous experiences (Mezirow, 2009). It is this type of learning experience that we are particularly interested in - both designing stimuli for it, and the ways that meaning schemes are transformed as a result. Given the evidence that undergraduates often have a view of function that is limited to algebraic expressions and their associated graphs (e.g., Carlson 1998; Even, 1990) and that such understandings typically result in a "vertical line test" related definition of function (e.g., Carlson, 1998), we designed an experience that would problematize these understandings, thereby creating a stimulus for transformation.

One strategy that has been suggested for resolving common misunderstandings related to function is the use of a function machine as a cognitive root. The idea of a *cognitive root* was introduced by Tall and colleagues as an "anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built" (Tall et al., 2000, p.497). As an example of a cognitive root for function concepts, Tall et al. suggest the use of a *function machine* (sometimes referred to as a function box). The machine metaphor Tall and colleagues describe is typically a "guess my rule" activity where the inputs and associated outputs are provided and students are challenged to determine what happened in the function machine (i.e., determine the function rule). While students are presented with a machine to embody the function concept, the rules used by the machine are algebraic in nature. In their studies using such machines proved quite promising as a cognitive root for function, yet some students still struggled with connecting representations and determining what is and is not a function (McGowan et al., 2000). Given the promise of a machine metaphor as a cognitive root for function coupled with our desire to present a disorienting dilemma for undergraduates, we set out to design an applet as a learning experience.

Design of the Applet

The Vending Machine applet (version 2.0) was designed to trigger a disorienting dilemma in students' understanding of function. The applet contains no numerical or algebraic expressions, but instead was built on the metaphor of a vending machine. Our Vending Machine applet (https://ggbm.at/qxQQQ7GP) is a GeoGebra book consisting of four pages. The first two pages contain two soda vending machines each with buttons for: Red Cola, Diet Blue, Silver Mist, and Green Dew. When the user presses a button (input), one or more cans appear in the bottom of the machine (output). To remove the can(s) from the bottom of the machine, the user clicks the "take can" button. On each of the first two pages, one machine is labeled as a function and the other is labeled as not a function. The non-function machines each have at least one button that produces a random can when pressed (i.e., the resulting can is not predictable based upon the button that is pressed). The directions ask the user to explore Machines 1-4 on Pages 1 and 2 and make a conjecture about why Machines 1 and 3 are functions and Machines 2 and 4 are non-functions.

Pages 3 and 4 of the applet allow the user to test and adapt their conjecture through interacting with 10 additional vending machines, Machines A through J. The functionality of each machine was designed to address misconceptions from the literature on distinguishing functions and non-functions. Examining Table 1, you will notice that the understandings we are trying to disrupt are the notion of what represents an element in the range (Machines B, I, & J), students occasional use of the term "unique" when thinking about outputs (Machines B & I), and the notion that functions should be "predictable" (Machines A, C, I, & J) - meaning that if one knows the function rule and is given an output, it is possible to determine what input resulted in that output.

Table 1. Machine output for each button pressed

		Button Pressed		
	Red Cola	Diet Blue	Silver Mist	Green Dew
Machine A	red can	blue can	silver can	random can
Machine B	two silver cans	green can	red can	blue can
Machine C	random can	random can	random can	random can
Machine D	silver can	green can	red can	blue can
Machine E	red can	silver can	silver can	green can
Machine F	blue can	silver can	green can	red can
Machine G	green can	green can	green can	green can
Machine H	red can	red can	silver can	silver can
Machine I	random pair	blue can	silver can	green can
Machine J	red can	blue & random can	silver can	green can

Method

The purpose of this study was to examine the understandings of functions that undergraduate students developed while engaging with the Vending Machine applet. We specifically address the following research questions: 1) How do students define function? and 2) How do students change their definition of function as a result of engaging with the Vending Machine applet?

Data Collection

A total of 123 students at six post-secondary institutions, that ranged in size, location, and focus, participated in the study. These students were undergraduates who had completed Calculus I. Prior to their use of the applet, students were asked to write a definition of function in their own words based on their current understanding, i.e., a pre-definition. This was done toward the beginning of the course or before functions were discussed, and no explicit instruction or discussion of functions had yet occurred. This data was collected by the instructor and students subsequently engaged with the Vending Machine applet outside of class, recording their interaction via a screencast. During the following class session students were asked to define function once again, a post-definition. The data used for this particular paper are the students' pre-and post- definitions.

Data Coding and Analysis

Given that our goal was to make sense of students meaning schemes within their written responses (i.e., text data) the use of content analysis (Creswell, 2007) as an analysis technique is appropriate. Specifically, we used *directed content analysis*. Directed content analysis uses existing theory or prior research to identify key concepts as initial coding categories for recognizing patterns in text responses (Hsieh & Shannon, 2005). For example, one set of initial codes were defined based on prior research that has identified student conceptions of functions as objects (e.g., expressions, graphs, tables of values) and relationships (e.g., mappings) (e.g., Dubinsky & Harel, 1992; Breidenbach et al., 1992). Through open coding, additional codes were defined as they emerged from the data. For example, students often included examples within their definitions so we created a set of codes to capture the nature of these examples (e.g., graph example, expression example).

Our completed codebook included 16 codes. To establish reliability in our coding a subset of 25 randomly selected definitions were coded independently by all six members of the research team and the number of agreements were divided by the number of assigned codes. The team had 93.1% agreement, so the codebook was considered reliable (Miles & Huberman, 1994). Once reliability was established, definitions were coded independently by six coders, with all definitions double-coded by pairs of coders. Pairs then compared codes and discussed and resolved differences (DeCuir-Gunby, Marshall, & McCulloch, 2011).

Prior to attending to aspects of the text individually, the pre- and post-definition were taken as a whole and coded in terms of their *accuracy* (i.e., correct, incorrect, or close to correct). Key elements of a correct definition were 1) the definition was not limited to a specific type of function (e.g. linear or quadratic), or to a particular representation (e.g., equation), and 2) the definition addressed the idea that functions map each input to one and only one output. Definitions coded as close to correct included those that indicated each input has one and only one output, but were not classified as correct because they were not general enough (e.g., the definition limited a function to a particular representation, such as an equation).

Next, each definition was coded regarding whether the definition indicated a function was a relationship (e.g., mapping), an object (e.g., equation, graph), or neither (see Table 2). We referred to this set of codes as *focus*, as they indicated how the students "saw" function. This coding was intended to be mutually exclusive, although some exceptions were found. Finally, definitions were coded according to whether or not they attended to output, as this was another aspect of the definition that we expected to be problematic based on the literature (Carlson, 1998; Even, 1993). After coding was completed, results for each code were summarized and analyzed for patterns and themes that provided insight to transformations of students' meaning schemes related to the definition of function.

Results

Pre-definitions

Our results show that the vast majority of students initially incorrectly defined function and their definition focused on a function as an object (Table 2). Only 24 students (19.4%) defined function correctly or close to correct. Note that with respect to focus there is a fourth category, *both object and relationship.* This is because, although the categories of relationship, object, and neither were meant to be mutually exclusive, there were definitions that referred to a function as both a relationship and an object. For example, one student defined a function as "an expression that is representative of a relationship between two or more variables." Referring to a function as an expression would generally be categorized as an object, but in this case the student also refers to a function as a relationship. Finally, there were initially 47 students (38.2%) who paid attention to output in their definitions. For example, "for each value of x, there can only be one and only one y" and "function is when there is a specific output given an input" were both coded as "attention to output". Definitions that made mention of the vertical line test were also coded as paying attention to output.

Table 2. Code occurrences n pre- and post- definitions

Code	Example	Pre (%)	Post (%)
	Accuracy		
Correct	A function is a relation in which for every input there exists exactly one output.	6 (4.8)	8 (6.5)
Close to Correct	A function is a mathematical equation in which a single input only yields one result.	18 (14.6)	46 (37.4)
Incorrect	An expression that is representative of a relationship between 2 or more variables	99 (80.5)	69 (56.1)
	Focus		
Relationship	A mapping from a domain to a codomain (or range)	14 (11.4)	18 (14.6)
Object	An equation with an x-input that gives a y output.	86 (70.0)	64 (52.0)
Both object & relationship	An expression that is representative of a relationship between two or more variables	7 (5.7)	10 (8.1)
Neither	f(x) → y; i) unique y value for every x; ii) one to one	16 (13.0)	31 (25.2)

Changes in definition

In this section, we report results in terms of how students' definitions changed after interacting with the applet. In terms of accuracy, the majority of students (52.8%) persisted in an incorrect definition of function. However, considering students who moved from incorrect definitions to definitions that were either correct or close to correct, over one-fourth (27.6%) of students improved their definition of a function. Furthermore, only 4% of students regressed in their understanding of function, i.e., from correct or close to correct to incorrect, or from correct to close to correct.

To better understand the aspects of function that are still problematic in definitions we look at focus and attention to output. In terms of students' focus in their definitions of function, a total of 90 students (73.2%) did not change (Table 3). Furthermore, the majority of students (69.1%) started with a definition of function that was classified as an object, and most (52%) persisted in that view. In terms of students' attention to output, 45 of the 47 students who initially made special reference to the output of a function maintained a focus on output in their definitions after engaging with the applet. On the other hand, of the 76 students who did not make any special reference to output in their initial definitions, 60 (78.95%) of them did so after engaging with the applet. Overall, 85.3% of all students attended to output in their revised definition.

Table 3. Occurrences of pre and post definition characteristics

		Post-Definition Characteristics								
		Incorrect	Close to correct	Correct	Object	Relationship	Both O & R	Neither	Attend to output	No attention to output
Pre-Definition Characteristics	Incorrect	65 (52.8)	31 (25.2)	3 (2.4)						
	Close to Correct	3 (2.4)	14 (11.4)	1 (0.8)						
	Correct	1 (0.8)	1 (0.8)	4 (3.3)						
	Object				64 (52.0)	8 (6.5)	1 (0.8)	12 (9.8)		
	Relationship				1 (0.8)	10 (8.1)	0 (0.0)	6 (4.9)		
	Both O & R				0 (0.0)	1 (0.8)	5 (4.1)	1 (0.8)		
	Neither				3 (2.4)	0 (0.8)	0 (0.0)	11 (7.9)		
	Attend to Output								45 (36.6)	2 (1.6)
	No attention to output								60 (48.7)	16 (13.0)

Discussion

With the essential role function plays in college mathematics, it is imperative that students have an understanding of function beyond an algebraic understanding. To address this, we created an applet, building from Tall and colleagues (2000) suggestion of a function machine as a cognitive root. Our Vending Machine applet (version 2.0) was designed to provoke disorientating dilemmas related to students' understanding of function which promote reflection and ideally shift students' meaning schemes related to definition of function away from an algebraic view. Since little research has been conducted on undergraduates' definition of function since Vinner & Dreyfus (1989), one of the goals of the study was to examine the current definitions of a function from a large sample of undergraduate students from six universities. From examination of students' definitions before engaging with the applet, our results showed that only approximately 5% of students could correctly define a function. The majority of students in this study did not include in their definitions the two key elements that define a function: 1) it is applicable across different representations; and 2) functions map each input to one and only one output.

From examining changes in students' definitions, the Vending Machine applet seemed to support students in moving toward a correct definition of function, and promoted greater awareness of the importance of the output in relation to the definition of function. However, since 55.2% of the students' post-definitions were limited to a specific representation (i.e., graph, equation) the applet did not seem cause a dilemma for students to move away from their algebraic view of functions in their definitions. An examination of students' screencasts (see Martin, Soled, Lovett, & Dick., under review) showed that students did not experience the dilemmas we had designed for as they worked through the pages of the applet. The first two pages of the applet, that told students which machine was a function and not a function, seemed to cause students to only used what they learned from these machines when determining if the other ten machines were a function or non-function.

Even though the Vending Machine applet (version 2.0) seemed to help students move towards a correct definition of function, for many students the experience did not provoke a dilemma regarding function as an object (e.g., equation, graph). It is possible that the ways in which the students interacted with the applet might have prevented the dilemma from occurring. Thus, we felt the applet could be improved, and as such the applet has been revised. Version 3.0 now consists of four pages in a slightly different format and includes two new machines. The first three pages each contain two vending machines (similar to version 2.0) except the directions say "Which one is a function?" Our intent is that this version will allow students to deepen their understanding of function by applying their knowledge of the function concept to the vending machines instead of using their experiences with the first four machines in version 2.0 to determine whether the other machines are functions or non-functions.

Conclusion

The results of this study suggest that the Vending Machine applet has the potential to be a powerful tool (cognitive root) for disrupting students' limited view of function and supporting their transformation of meaning schemes related to the concept of function. However, to determine whether or not these findings are generalizable, the use of the newest version of the applet (3.0) needs to be studied on a larger scale. Our plan is that through further study and revision, we will produce a transformative cognitive root that disorients students' algebraic view of function, remedies existing misconceptions, and on which conceptual understanding of function concept can be built.

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