Pedagogical Considerations in the Selection of Examples for Definitions in Real Analysis

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This study investigates mathematicians' pedagogical practices and associated beliefs about the use of examples to instantiate definitions in a real analysis textbook. We used task-based interviews, asking participants to revise the introductory presentation of a concept, including definitions and examples, to be of higher pedagogical quality. All mathematicians believed that examples and counter-examples are important in learning about a concept. In this report, we concentrate on how mathematicians take the collection of examples and student thinking into account when deciding on which examples to use and the types of criteria they use to determine an appropriate collection of examples for a definition.

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Leinhardt, Zazlavsky, and Stein wrote that "A primary feature of explanations is the use of well-constructed examples, examples that make the point but limit the generalization, examples that are balanced by non- or counter-cases" (1990, p. 6). Similarly, researchers have asserted that "exemplification is a critical feature in all kinds of teaching, with all kinds of mathematical knowledge as an aim" (Bills & Watson, 2008, p. 77). This study explores the use of examples as a part of pedagogical practice in proof-based courses, and, in particular, real analysis. In these courses, one important way of presenting mathematical subject matter is via examples. In particular, a recent study of 11 proof-based undergraduate mathematics lectures (each between 60 and 75 minutes) included, among the findings, that 65 examples were presented across the lectures, with every professor discussing at least one example, and the median professor discussed 5 examples during a single lecture. That is, the presentation of examples appears to be a common part of the pedagogical practice of mathematicians while giving instruction about proof-based mathematics.

The Pedagogical Importance of Examples

Authors have claimed that examples are important in developing conceptual understanding (Mason & Watson, 2008; Vinner, 1991) and knowledge and use of examples is a mark of expertise in mathematics (Michener, 1978). Examples have been claimed to help students develop understanding of mathematical definitions (Antonini, 2006; Leinhardt, Zazlavsky, & Stein, 1990), and, examples can help students interpret, create, and prove mathematical theorems (c.f., Cuoco, Goldenberg and Mark, 1996; Lakatos, 1976). As part of the theorem generalization process, examples have been described as essential for generalization and abstraction (Antonini, et al, 2011). The perceived pedagogical power of examples (Antonini, et al, 2011; Bills & Watson, 2008; Mason & Watson, 2008) has led to, among others, the exploration of graduate students' use of examples to determine the truth of conjectures (Alcock & Inglis, 2008) and of the principles K-12 teachers use in selecting examples to use with their students (Rowland, 2008; Zodik & Zaslavsky 2008). For example, teachers use examples to motivate basic intuitions or claims about new material (Michener, 1978). Similarly, there is evidence that asking students to generate boundary examples can help clarify the need for criteria in a definition or hypotheses in a proof (Mason & Watson, 2001). Interviews with mathematicians suggest that they also attribute some of these pedagogical values to examples (cf. Alcock, 2010; Michener, 1978;

Weber & Mejia-Ramos, 2011; Weber, 2012). More, they report using examples as part of their presentation of proofs (Alcock, 2010; Weber, 2012), and to instantiate claims or definitions (Alcock, 2010). Observational studies provide evidence that these claims are representative of their pedagogical practice (c.f., Fukawa-Connelly & Newton, 2014; Mills, 2014).

Mills (2014) observed four mathematicians teaching advanced mathematics courses and found that they used examples to motivate the statement of a theorem, instantiate a concept, or illustrate results. She did not describe or classify which examples the instructors used or their associated rationale for these choices. Fukawa-Connelly and Newton (2014) provided some insight in this regard by investigating one professor's use of examples of the concept of group in an abstract algebra class, drawing on the notion of the enacted example-space. They found that relatively few examples were made part of the content of the class, but that each of them was used repeatedly. Finally, Cook and Fukawa-Connelly (2015) surveyed and interviewed algebraists about the examples of groups and rings that the believed to be most important for students to know at the end of an introductory group theory course. They found that algebraists typically named classes of groups (e.g., the cyclic groups) rather than concrete examples and that there was relatively little consensus about the set of examples that students should know. The rationales that the mathematicians provided for their choices focused on the familiarity and ease of instruction about examples (using words like *simple* and *nice*), the historical foundations of the subject, the ability to demonstrate different ideas and concepts (including helping students avoid inappropriate generalizations), those that allow helpful visual representations, and experience the prevalence and variety of groups and rings. Finally, we note that Peled and Zazlavsky (1997) differentiated between different types of counter-examples frequently used by mathematics teachers. The described specific (e.g., the integers) that can show a statement is not true, and general (e.g., where one length in a figure might be a variable) that can help explain why a statement is not true.

The current study build on the literature in two important ways. First, we note that only two of these studies identify specific examples that mathematicians use in their teaching, and only one study directly reports on the corresponding rationale for the mathematician's choice of examples. Yet, that study explored examples at the most general level; asking about the entire collection of groups and rings that students should know *at the end of a course*, rather than exploring the examples used to explain what a group or ring is to a student. While Fukawa-Connelly and Newton (2014) examined the examples used to instruct students on the concept of a group, they did not interview the professor and so were unable to provide any of the instructor's rationale for his pedagogical decisions. We found evidence that mathematicians attend to several different categories of information when considering the examples in their text: individual examples and their properties, the connection of examples and its properties, what should be explicit in the text and how/where is should appear, and aspects of student (or reader) thinking and their relation to the examples and concepts. In this paper we advance two claims:

1. Mathematicians attend to properties of the collection of examples: the size, diversity, and ordering of this collection as well as whether it has duplicates.
2. Mathematicians attend to the aspects of student/reader thinking while working with the text: interaction with prior knowledge, intuition/informality and expected lack of intuition, and teaching general cognitive skills.

Methodology

Rationale

In this study, mathematicians were given four (researcher-created) introductions to concepts from real analysis textbooks, each of which included a definition statement (or more than one) and may have included examples and discussion of the concept. The mathematicians were instructed to revise these introductions to improve their pedagogical quality. We assert that the mathematicians' pedagogical thinking and values can be observed through the additions to and deletions from the text, their evaluations of the elements present in the text, and the rationales they give for their revisions.

This approach can show strong evidence that the mathematicians do care about an aspect of an introduction, but their silence on an aspect is not evidence that it is unimportant to them. Because of this asymmetry, we designed the introductions to be diverse with respect to all aspects that we identified in our literature review (including the types of revisions that Lai, Weber, and Mejia-Ramos (2012) identified), pilot interviews, and personal teaching experience, including the presence and number of examples, formality and abstraction, motivation, precision, and presence of normative notation.

We used participants' revisions, evaluations, and rationales for the revisions to form hypotheses about what they believed a good pedagogical introduction to a concept should include, with a focus on what the appropriate exemplification of a definition would be. We note that the exemplification appropriate for a text may not be appropriate for a lecture and we do not intend our work to make any claims about exemplification in lecture.

Method

Participants. The first author invited mathematicians to participate in the study. He solicited the participation of 10 mathematicians, and we do not have any a prior reason to believe these participants more interested in or capable at mathematics teaching than other mathematicians. The research expertise of the participants included analysis, applied math, functional analysis, analytic number theory, and geometric topology and their teaching experience ranged from two years as a graduate teaching assistant to over twenty years as a teaching-focused institution. All of them had taught or were preparing to teach real analysis. We assigned all mathematicians a single-initial designator (that is not either of their initials) and refer to them with gender-neutral pronouns in order to protect their anonymity.

Materials and Procedures. Each participant met individually with the first author for a task-based interview. Participants were presented with four revision tasks sequentially. For each, they were given a 1-page, complete textbook introduction to a concept that we believed to be mathematically correct, told that the target audience for the text was a student in a junior-level real analysis course, and asked to revise it to improve its pedagogical quality. After they finished revising each introduction, they were asked to describe each change and their rationale for it, about broad categories of changes if they did not make them (including changing the collection of examples), about the aspects of the introduction they left unchanged, whether anything in the introduction was atypical, about their goals, and whether any of their comments would have been different in some medium other than a textbook.

Participants were told to "think aloud" as they were making their revisions. Because this task asked participants to improve the pedagogical presentation of the definition and because we explained that it would be for a textbook for a specific undergraduate course, we assumed the participants would treat these as pedagogical presentations. At times, we did need to clarify that the presented materials constituted the whole of the presentation of the definition and that the next items would be propositions and proofs. Each interview was audio-recorded and subsequently transcribed, and participants were asked to make any needed written changes (although they often specified multiple changes aloud while writing relatively few in comparison); any written productions were subsequently scanned.

Analysis. To analyze the types of revisions performed, we used an open coding scheme in the style of Strauss and Corbin (1990). While we were sensitive to the prior literature, because none of it related to textbook authorship and none focused on real analysis, we believed that an open-coding scheme would be more appropriate. For each revision (including purely evaluative statements) that a participant made on the revision tasks, we made a general description of the edit and what aspect of the presentation it was describing and used these aspects as category names. We then went through the transcript and noted the reason given for the edit (if any). For the purposes of this study, we then collected all revisions related to examples, and ignored those about other aspects of the introduction. We again engaged in another round of open coding, developing categories of codes that described any revisions and related codes for the rationales that participants provided. As appropriate, we coded new instances using categories that we had already developed or created new categories as needed. As we coded, we continually refined our coding manual, including revising names and definitions of categories, and, noting which sets of categories were orthogonal and which were overlapping, such that a particular instance should not carry codes from both categories. Once the categories were formed, we recoded all the data and resolved any remaining issues through discussion.

Data and Results

All of the mathematicians indicated that they believe that examples should be part of the pedagogical presentation of a definition. For example, on Definition 4, we presented the definition without any examples and all of them mathematicians indicated that their revision would include examples. Some suggested that 5 was the appropriate number of examples to include with the pedagogical presentation of a definition. Their descriptions of their goals for examples suggested that they were thoughtful about which examples would be included, the collection and sequencing of the examples, the relationship between the examples and the text, and, the range of examples presented. We chose to highlight 2 primary findings from these interviews that illustrate the types of thinking that mathematicians exhibited with respect to examples, first quantitatively then with specific quotes.

The interviews produced 184 distinct comments about examples to be coded, with a total of 626 codes assigned. Of these 116 codes were given for comments about individual examples, 62 about the collection of examples, 74 about the text and its explicit elements, and 45 about student thinking. These data support the claim that the mathematicians attend to all four of these aspects of the examples in a textbook introduction to a concept.

We have asserted more specifically that mathematicians attend to properties of the collection of examples: the size, diversity, and ordering of this collection as well as whether it has duplicates. The code Collection and its subcodes were present in the comments for all four definition tasks (7,17,12,26) for a total of 62 coded items. Similarly, Collection and its subcodes were present in the comments from all ten participant interviews (7,6,5,6,8,7,5,4,5,9). We have also asserted that mathematicians attend to the aspects of student/reader thinking while working with the text: interaction with prior knowledge, intuition/informality and expected lack of intuition, and teaching general cognitive skills. Thinking and its subcodes were present in all four definition tasks (14,9,9,13) for a total of 45 coded items. Similarly, Thinking and its subcodes were present in the comments from all ten participant interviews (3,5,4,4,2,2,3,9,9,4).

Collection of Examples

All of the mathematicians attended to both individual examples and the collection of examples. The most common subcode of Collection was Diversity, which captures the participants' comments that the collection has or should have individual examples with different properties. For example, they valued having both examples and non-examples, "extremes of behavior", simple and complex examples, and various representations.

In this subsection, we illustrate how the professors claimed that examples should help students make sense of what the concept is. Dana, in discussing Definition 1, was very explicit in describing the types of thinking that examples needed to support:

> I'm trying to head off any confusion about exactly what the definition is of the increasing function. I'm trying to expose students to kind of broaden their universe in their head of what mathematical functions are. They're not just the functions that you differentiate in your calculus class. They can include functions that are not smooth and aren't defined everywhere. … to familiarize them with different types of examples.

We interpreted Dana as claiming that examples serve a number of roles in helping students come to understand a concept. First, that students might be confused about a concept and that examples can mitigate that. Second, that examples can force students to consider unfamiliar instances of known concepts, to "broaden their universe," when considering special classes of the previously known concept. In doing so, there are particular types of variation that professors might attend to, for functions that might be "not smooth" or "not defined everywhere." While not every professor was as explicit as Dana, they all indicated the importance of examples in helping students broaden their collection of examples to include more 'exotic' (which they often described as unfamiliar or complex) cases of common concepts.

We further illustrate how professors attended to how examples (including counter-examples) might support students in interpreting concepts. The first important way that professors believed examples can support student understanding of the definition of a concept is by helping to interpret the concept and distinguish it from other, similar, concepts (all of the professors made comments indicating that this was a consideration). For example, many of the professors made explicit statements about the value of counter-examples. For example, Kai claimed, "having counter-examples is just as important as having examples, knowing what something is and what isn't." Similarly, Cody claimed, "I often say to my students, "For every new definition we learn, we want to think of "an example but we also want to think of a counter-example." Cody then provided a specific instance, related to Definition 1, for which a counter-example would be helpful, "Yeah. Also find a function which is not increasing or strictly increasing." We interpreted Cody's statement as claiming that a function that is not increasing or strictly increasing would help students to understand how Definition 1, strictly increasing, is distinct from increasing or not-increasing, that is, the counter example could help illuminate the meaning of the inequalities in the statement of Definition 1. Brett explained why counter-examples are helpful, "If you just have a positive example of something, that doesn't help you really compare. Unless you have both a positive and a negative example it's hard to use that." We interpreted Brett as claiming that comparing examples with different features, especially one that has all of the needed features of a concept with one that does not, is one way that a learner might come to understand a concept, and, without a counter-example among the collection of examples, students would be unable to make such comparisons.

Similarly, the professors claimed that examples allow students to distinguish a concept from other, similar concepts, such as between a lower bound and a greatest lower bound. The idea of using examples to help students distinguishing between similar concepts was mentioned by all the professors at least once and at least once during the discussion of each definition. We illustrate this with two representative quotes of professor's the claims about using examples to differentiate concepts. For example, in discussing Definition 2 (lower bound and greatest lower bound) Kai claimed:

> I would add one more example, or I would modify an example. There is an example of the infimum, there are examples of lower bounds or examples of not lower bounds but in example 1, I would add that it is a lower bound but it is not the greatest lower bound because that thing gives counter-example to what an infimum is. It's a lower bound but not the greatest lower bound because –pi/2 is a greater lower bound; so I would add that.

In this case, the professor is asking to add or modify an example so that one of the examples will be a lower bound but not a greatest lower bound. Brett asked for a similar revision, "You can probably say it's something like -pi is a lower bound but not an infimum, or you can say that …" Both of these professors were using examples to illustrate the difference between two, similar, concepts.

Second, the professors were very explicit that they attended to whether individual examples had properties that were not required by the definition, and, when all of the examples in the collection had the same extra properties. That is, the professors attended to whether examples had unnecessary properties. In discussing the definition of a convergent sequence, Morgan noted, "Note that the limit doesn't have to be attained though it can be and I will probably give an example of that." This comment specifically notes a property that the definition of convergent does not require, that the limit be attained, and that the professor thinks that a good pedagogical presentation would include examples with both cases. Similarly, Harper noted that, "I don't particularly see it in the numerical example that they give, but the pictorial example includes it, meaning that the sequence doesn't have to be on one side of the limit. I believe the numerical examples that I see right now, all of them stay on one side of the limit, but the diagram does not, so that is fine." Our interpretation of Harper's comment is that the definition of a sequence does not require that the sequence "stay on one side of the limit," meaning that this is a property that is not required. Second, Harper attended to whether each of the examples had this property, evaluating each of the numerical examples and the diagram, and noting that in the collection, at least one example does not have the unneeded property. Similarly, Dana gave a positive evaluation of an example illustrating a property of infima, "I like that example, the fifth one, because it shows that the infimum may be a member of the set or it might not. Either one is possible. I like this example, the set's an interval." Here, we interpreted Dana's evaluation as first stating the unnecessary property, whether the infimum is a member of the set, and then giving a positive evaluation of the fact that the collection of examples includes at least one example that does not have that property. In the presentation, we will further support these claims and illustrate additional ways that the professors believed that examples could support student understanding of a concept.

Student Thinking

All of the mathematicians attended to student thinking in relation to the examples. The majority of the codes in this category were for Thinking rather than a subcode, indicating either that there were diverse aspects of thinking under consideration that did not come together into subcodes or that the participants most often talked about student thinking in the context of the particular concept rather than general principles about thinking. The most common subcodes captured participants' comments that examples would or should be familiar or unfamiliar to students, that working with examples in the context of a particular example taught a general skill, and that students would or would not have intuition about the examples. While the participants valued building intuition for the concepts, they also valued non-obvious examples about which students would not have intuitive conjectures or about which their intuition would be wrong because these both generate pedagogical situations that are useful for teaching students to work carefully with definitions and proofs.

For real analysis, the most common source of salient prior knowledge was calculus. Jesse observes that "analysis is trying to formalize calculus. An increasing function is something that they've seen in calculus a lot. They've never really talked too much about lower bounds and greatest lower bounds in calculus." Moving beyond description of familiarity for students, Morgan suggests revising a scatterplot to the graph of a familiar function:

> I guess I'm thinking of taking a standard graph of 1/x and thinking of what the infimum of the values of the function are. Which is sort of related to what this example was going to do but it seems to me that it's again this is a graph that they have seen before however I think that it is ... Would have presented it in a context or something that they had not thought before. Actually, they have thought about asymptotes so maybe it would be a way to relate this new concept with something that they have seen before."

Conversely, Kai suggests that "Examples that they have not seen before or wouldn't necessarily come up with on their own, I think, would be really good for them to see."

The goal, according to Harper is "to make students understand the definition as deep as they possibly can, hence the types of examples that I provide. That's the nature of how I studied things back when I was a student. It's like looking for things in the definition which are weird, which are non-intuitive yet are included in this definition." And for Kai, "learning how to be very intentional in applying the language and notation of the definition, I think, is a good skill." Examples "help students see the pathology of things that can happen" (Jesse).

Discussion

This study has made two main contributions. First, it provides a fine-grained analysis of the factors that instructors consider when selecting examples to instantiate a particular definition as part of a textbook. The instructors all claimed that examples (and non-examples) are important in helping students understand a definition, and claimed that their goals in example selection are aimed at exactly this; helping students understand the particular definition. Their descriptions of their goals for examples suggested that they were thoughtful about which examples would be included, the collection and sequencing of the examples, the relationship between the examples and the text, and, the range of examples presented. In doing so, it provided further evidence that mathematicians are thoughtful about their instruction. More, one of the criteria that we described show that mathematicians take student thinking into consideration in their choices of examples; attempting to avoid common errors, 'head off' inappropriate overgeneralizations such as by ensuring that the collection of examples does not all share a particular unneeded property, and support the construction of a rich example space. Due to limited space, we have only been able to discuss two of our categories of codes and those two without much detail. We will develop these and other themes in more detail and show more data in a presentation in at the conference.

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