Figurative Thought and a Student's Reasoning About "Amounts" of Change

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This paper discusses a student coordinating changes in covarying quantities. We adapt Piaget's constructs of figurative and operative thought to describe her partitioning activity in terms of the extent that it is constrained to carrying out particular sensorimotor actions on perceptually available material, and we relate such descriptions to her thinking about quantitative amounts of change. We conclude the paper by discussing how characterizing these nuances of student thinking in terms of figurative and operative thought contributes to current literature on covariational reasoning and conceptualization of concept construction.

Keywords: Cognition, Piaget, Covariational Reasoning, Amount of Change

Researchers have shown that students' *quantitative* and *covariational reasoning*—the mental actions involved in conceiving measurable attributes changing in tandem (Carlson, Jacobs, Coe, & Hsu, 2002; Thompson, 2011)—are critical for their learning of function and rate of change (Ellis, 2011; Johnson, 2015; Thompson & Carlson, 2017). Stemming from the complexities of students' thinking, these researchers have called for investigations that identify nuances in students' covariational reasoning. We answer these researchers' calls by drawing on Carlson et al. (2002) and Saldanha and Thompson's (1998) notions of covariation to characterize a student's reasoning about amounts of change in various contexts. We extend extant literature using Piaget's (2001) notions of *figurative* and *operative* thought to explain the extent a student's reasoning was constrained to sensorimotor actions and perceptual results from those actions.

Quantitative Reasoning, Covariational Reasoning, and Partitioning Activity

Thompson (2011) described that the mental construction of a *quantity* involves "conceptualizing an object and an attribute of it so that the attribute has a unit of measure" (p. 37). Despite Thompson's use of "measure," he emphasized that reasoning about a specified quantity's value is unnecessary when reasoning quantitatively; sophisticated conceptions of quantity entail reasoning about a quantity's magnitude (i.e., amount-ness) while anticipating that it has an infinite number of measure-unit pairs (Thompson, Carlson, Byerley, & Hatfield, 2014). Such distinction between a quantity's magnitude and its measures enables us to account for reasoning about covarying quantities that is not constrained to the availability of values; focusing on a quantity's magnitude affords characterizing mental activity in terms of perceptual material associated with a quantity's amount-ness (e.g., a segment that represents a quantity of distance).

An individual imagining variations in a quantity's magnitude (and hence value) is positioned to reason covariationally. When reasoning covariationlly, "a person holds in mind a sustained image of two quantities' values (or magnitudes) simultaneously…one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value" (Saldanha & Thompson, 1998, p. 299). Building on Saldanha and Thompson's (1998) covariation, Carlson et al. (2002) specified mental actions involved in coordinating quantities, among which students' coordination of amounts of change of one quantity with respect to changes in another (Mental Action 3 in their framework) is central to our work here. An individual coordinating amounts of change imagines quantities' magnitudes accumulating in successive states (and possibly anticipates continuous covariation between these states; see Thompson and Carlson (2017)). To illustrate, a student reasoning about covarying quantities B and K can envision the magnitude ||B|| accumulating in equal accruals, construct the accumulation of ||K|| in terms of corresponding accruals, and coordinate those accruals in ||K|| to conceive ||K|| increasing by decreasing amounts with respect to ||B|| (see Figure 1a-c for an illustration with respect to the Taking a Ride task in Figure 3a). Because coordinating amounts of change involves the activity of constructing a magnitude's accumulation in terms of accruals, we use *partitioning activity* to refer to students' mental and sensorimotor actions associated with their producing and reasoning about these increments that may represent amounts of change.

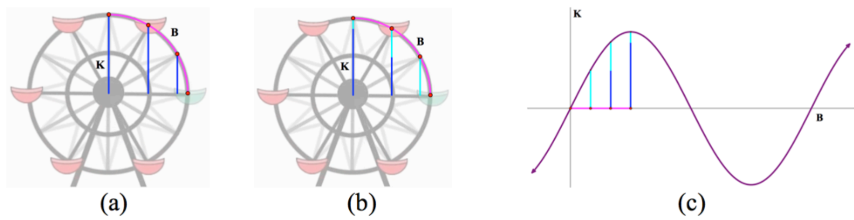


Figure 1. As quantity B increases by equal amounts (denoted in pink), quantity K increases (denoted in dark blue, (a)) by decreasing amounts (denoted in light blue, (b)), which can be represented in a Cartesian system (c).

Figurative and Operative Thought

Table 1. Figurative and Operative Partitioning Activity

Partitioning Activity	Foregrounded Actions of Partitioning Activity
Figurative Partitioning Activity	Repeating sensorimotor actions of partitioning tied to particular perceptual material and results; Potentially constrained to *available* perceptual material; Conceived invariance is with respect to sensorimotor actions and their perceptual results
Operative Partitioning Activity	Sensorimotor actions subordinate to mental actions (e.g., quantitative and covariational reasoning); Can anticipate partitioning activity on available or hypothetical perceptual material; Conceived invariance is with respect to coordinated mental actions and their transformations

We have found the theoretical distinction between *figurative* and *operative* thought (Piaget, 1976, 2001; Steffe, 1991; Thompson, 1985) useful in developing models of students' partitioning activity. Piaget (1976, 2001) characterized figurative thought as based in and constrained to sensorimotor actions and perception, and he described operative thought as the coordination of mental operations so that these coordinations dominate figurative material (i.e. sensorimotor actions and perceptual material). We emphasize that characterizing a student's thinking as operative does not imply her thinking does not entail fragments of figurative material. Likewise, characterizing a student's thinking as figurative does not imply that her thinking does not entail operative schemes. A researcher's sensitivity to these distinctions is an issue of "figure to ground" (Thompson, 1985, p. 195). When a student's thinking foregrounds carrying out repeatable (mental or sensorimotor) actions and the results of those actions, it is figurative; when a student's thinking foregrounds the coordination of actions and transformations of those actions and their results, it is operative. The issue of foregrounding is important for describing students' partitioning activity because such activity necessarily entails figurative material (e.g., drawing and producing graphs and partitions) and likely entails operative schemes (e.g., understanding a coordinate systems in terms of directed distances). In characterizing students' partitioning activity in terms of figurative or operative thought, we thus make the distinctions in Table 1.

To illustrate these distinctions, consider a student determining if the graphs in Figure 2a and Figure 2b represent the linear relationship $y = 3x$. With respect to Figure 2a, a student who engages in figurative partitioning activity could imagine the graph in terms of the successive movements of one axes mark to the right (denoted in blue) and then three axes marks up (denoted in red), and associate such movements with a positive slope (Paoletti, Stevens, & Moore, 2017). With respect to Figure 2b, the student could conceive movements to the right (denoted in blue) along the graph as corresponding to movements down the graph (denoted in red), and associate such movements with a negative slope. In each case, the student's thought is dominated by carrying out or repeating particular sensorimotor actions to the extent that associations (e.g., a line falling left-to-right necessarily has a negative slope) are tied to that activity and its results. Hence, the student concludes that the two graphs are different.

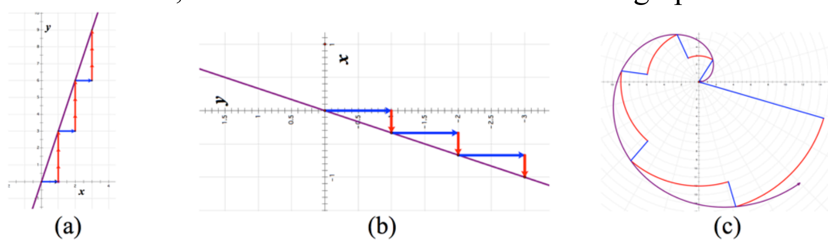


Figure 2. (a) A graph that represents the relationship of y=3x in a Cartesian coordinate system, (b) a rotated graph of (a), and (c) a graph that represents the relationship of r=3θ in a polar coordinate system.

In comparison, a student who engages in operative partitioning activity could conceive that both graphs are such that any directed change in x corresponds to a directed change in y three times as large as that in x. The student's partitioning activity is operative because she can coordinate and transform activity specific to each graph to conceive an underlying invariance that dominates figurative differences in activity. The student might anticipate re-presenting invariant partitioning activity in other contexts or coordinate systems (e.g., polar coordinates, Figure 2c). The anticipation of re-presenting partitioning activity aligns with Moore and Silverman's (2015) *abstracted quantitative structure*: a structure of related quantities a student has internalized as if it is independent of specific figurative material (i.e., representation free).

Methods

This paper reports results of a semester-long teaching experiment (Steffe & Thompson, 2000) with prospective secondary mathematics teachers (PSTs; Lydia, Emma, and Brian). They were in their first semester of a four-semester secondary mathematics education program at a large university in the southeast United States. We conducted 10-11 teaching sessions (1 to 2 hours each) with each PST. The project principal investigator (the second author) served as the teacher-researcher (TR) at every teaching session. At least one other research team member was present as the observer(s). Each session was videotaped and digitized for analysis. In both ongoing and retrospective analyses efforts, we conducted conceptual analysis (Thompson, 2008) to develop models of PSTs' mathematics. Specifically, our iterative analyses efforts involved constructing hypothetical mental actions that viably explained the PSTs' observable and audible behaviors. We continually searched the data for instances that the models could not account for, and we modified our models or we attempted to explain developmental shifts in a PST's meanings. In this paper, we focus on the case of Lydia because of particular aspects of her partitioning activity that were consistent throughout the teaching experiment. We consider it important to characterize her ways of thinking in order to add nuances to our prior conceptualizations of students' quantitative and covariational reasoning.

Task Design

We describe Lydia's activity on three related tasks: (1) Taking a Ride, (2) Which One, and (3) Circle. Taking a Ride included an animation of a Ferris wheel (Desmos, 2016) (see Figure 3a) and focused the students on constructing the covariational relationship between the *height* of the green rider above the horizontal diameter of the wheel and its *arc length* traveled (the sine relationship; Moore (2014)). Which One (Figure 3b) was presented after students' first encounter of Taking a Ride. It included a simplified version of a Ferris wheel (left) with the position of a rider indicated by a dynamic point. The topmost line segment (shown in blue, right) represented the arc length the rider had traveled counterclockwise from the three o'clock position. Students could vary the segment length by dragging its endpoint with the dynamic point on the circle moving correspondingly. We asked the student to determine which of the six red segments, if any, could accurately represent the rider's height above the horizontal diameter as the rider's arc length varied. Segment 1 is a normative solution and segments 2-6 vary with either different directions or rates. In students' initial attempt on these two tasks, we did not prompt them to graph because we wanted to gain insights into their reasoning with displayed magnitudes in contexts that minimized the influence of their previously constructed graphing meanings. For Circle (Figure 3c), we asked students to graph the relationship between the *horizontal distance* and the *arc length* associated with a dynamic point (i.e., the cosine relationship). Collectively, we designed the series of tasks to provide different figurative material to tease apart the extent that a student's reasoning was dominated by figurative or operative thought.

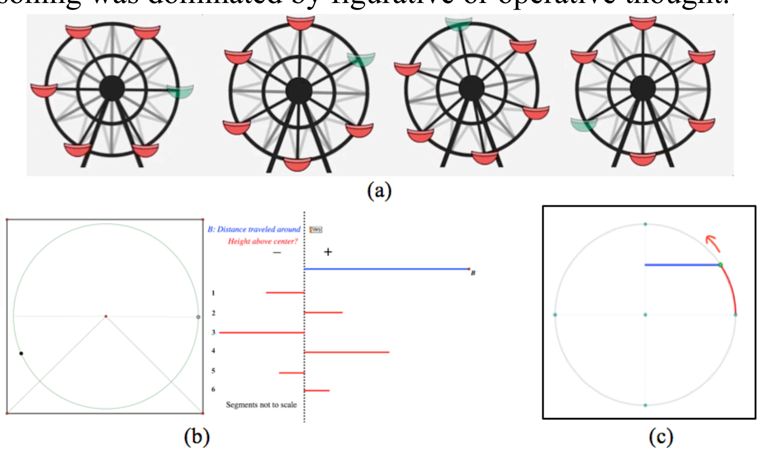


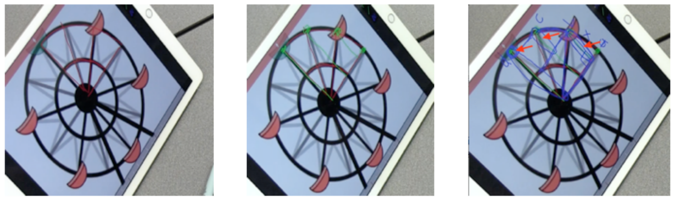
Figure 3. *(a) Snapshots of Taking a Ride, (b) Which One (with segment numbers labeled), and (c) Circle.*

Results

In this section, we illustrate Lydia's partitioning activity with a focus on the figurative material constituting her partitioning activity as she considered a variety of representations.

Re-presenting Partitioning Activity

In the first teaching session, we worked with Lydia on Taking a Ride (Figure 3a). With much effort, Lydia constructed what we perceive to be successive amounts of change of height for successive, equal changes in arc length (see her construction in Figure 4a-c). Noticing that the blue segments (in Figure 4c) decreased in magnitude, Lydia concluded that, "[A]s the arc length is increasing... [the] vertical distance from the center is increasing ... but the value that we're increasing by is decreasing." Suggesting she was excited that she had identified this relationship, she explained with enthusiasm, "I just discovered this by myself." This revealed that her activity of drawing partitions and identifying amounts of change was novel to her at the time.



(a) (b) (c)

Figure 4. Lydia (a) partitioned the distance traveled in equal increments, (b) identified height of the green rider in each successive state, and (c) identified amounts of change of height.

Immediately following this task, we presented the Which One task (Figure 3b). After some explorations, Lydia claimed that she would like to choose a red segment that is moving at a constant rate. She eliminated four of the six segments and had hard time deciding which of the other two segments was moving constantly (Figure 5a). She then decided to orient one of them (a normatively correct solution) vertically, and put it inside the circle (Figure 5b). She then confirmed that the length of that segment matched the height of the dynamic point for different states (Figure 5c). When asked if the segment entailed the amounts of change relationship constructed in the initial Taking a Ride task, she responded:

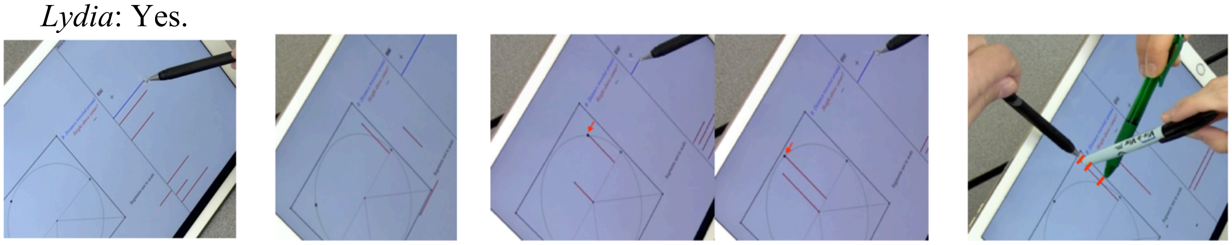
Lydia: Not really…Um, I don't know. [*laughs*] Because that was just like something that I had seen for the first time, so I don't know if that will like show in every other case…Well, for a theory to hold true, it like – it needs to be true in other occasions, um, unless defined to one occasion.

TR: So is what we're looking at right now different than what we were looking at with the Ferris wheel?

Lydia: No. It's – No…Because I saw what I saw, and I saw that difference in the Ferris wheel, but I don't see it here, and so –

TR: And by you don't see it here, you mean you don't see it in that red segment?

Lydia: Yes.



(a) (b) (c) (d)

Figure 5. (a) Lydia was working on the Which One task, (b) checking the red segment point-wisely, and (c) we were assisting Lydia to identify amounts of change in height.

Lydia described height increasing by decreasing amounts as a "theory" that needed to be tested in this new situation. Her knowing that the red segment worked for each state did not imply by necessity that the red and blue segments existed in a covariational relationship consistent with that between height and arc in Taking a Ride. Following this exchange, and after the researchers created perceptually available material by using pens to denote amounts of change of the red segment (Figure 5d), Lydia responded in surprise that her "theory" held true.

We characterize Lydia's partitioning activity as figurative due to her difficulty re-presenting such activity from one context to another. She identified successive height accruals on the Ferris wheel (Figure 4c), but her understandings of amounts of change were rooted in carrying out activity and creating perceptually available increments in that context. When moved to a context with magnitudes changing continuously, she did not anticipate or re-present her partitioning activity. That is, as she considered successive red segment states in Which One, she was unable to hold in mind the red segment associated with a prior state to compare it to a current state.

Situations, Graphs, and Figurative Material

The TR began the fourth session by asking Lydia what she recalled from the previous sessions, in which she worked on Taking a Ride (Figure 3a) and Circle tasks (Figure 3c). She started with drawing the first quarter of a circle (see Figure 6a for work):

> **"**So we kind of said as the arc length is increasing in the first quadrant that our X distance is decreasing [*drawing the horizontal segments within the circle from bottom to top in Figure 6a*], and then…distance will decrease more in the same amount of space. So like from here to here [*highlighting the bottom blue arc*], then we'll say these are the same arc length [*highlighting the top blue arc*]…so we're going to take this point here [*marking a point at the top of the far-right pink segment*] and then drag it down [*drawing the far-right pink segment*], we've only lost this much [*highlighting the shorter red segment*]. And then from here [*drawing the middle pink segment*] to here [*tracing the far-left pink segment*] we lost this distance [*highlighting the longer red segment*], but we're saying those are the same arc length [*pointing to the two blue arcs*], so it's a lot more distance."

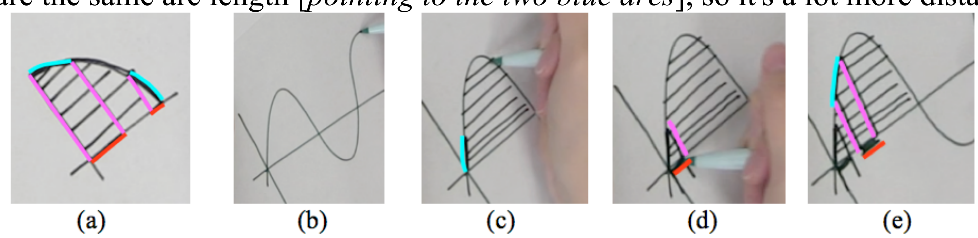


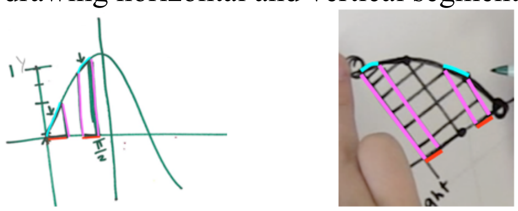
Figure 6. (a) Lydia's drawing of the circle situation, (b) Lydia's graph, and (c)-(e) her construction process.

Lydia's partitioning activity appeared compatible with that from previous sessions, and thus the TR asked Lydia how such activity related to graphing the relevant relationship. Lydia drew a graph (Figure 6b) and explained how the graph related to her partitioning activity in Figure 6a:

> "As we go up in arc length [*highlighting the blue curve in Figure 6c*]…that distance is decreasing [*drawing the horizontal segments from bottom to top in Figure 6c*], and so we see that here [*drawing the pink segment in Figure 6d*] is like this [*highlighting the red segment in Figure 6d*], and then [*highlighting the blue curve and drawing the pink segments in Figure 6e*] ... here is this [*drawing the red segment in Figure 6e*]. So that's the same conclusion we had gotten from the circle, so then we can say that this circle relates to this graph."

Lydia's partitioning activity across the situation and graph included: (a) drawing horizontal segments emanating from the circle and curve (see Figure 4a and 4c), (b) tracing arcs from lower end points to higher end points on the circle (denoted in blue, see Figure 6a) and tracing a curve on graph in the same manner (denoted in blue, see Figure 6c and 6e), (c) drawing vertical segments from the end points produced by the arcs and curve to a horizontal segment or line (denoted in pink, see Figure 6a, 6d and 6e), and (d) drawing horizontal segments between two pink segments and comparing their lengths (denoted in red, see Figure 6a, 6d, and 6e). We characterize Lydia's partitioning activity as figurative due to it foregrounding repeated sensorimotor actions that produce similar perceptual results (e.g., partitioning along something curved, drawing vertical segments, and drawing and comparing horizontal segments).

Providing additional evidence that Lydia's partitioning activity was figurative, later in the teaching session, Lydia drew a similar graph (Figure 7a) in order to discuss the relationship between “height” and “arc length”. Her activity included tracing from left to right two equal horizontal segments (denoted in red, Figure 7a), drawing vertical segments from end points of the vertical segments to the curve of her graph (denoted in pink, Figure 7a), and tracing two corresponding curves on her graph (denoted in blue, Figure 7a). She compared the lengths of these curves and concluded that the increases in height became smaller. Similarly, on a circle, she traced two horizontal segments (denoted in red, Figure 7b), drew vertical segments (denoted in pink, Figure 7b), and traced and compared two arcs on the circle (denoted with blue, Figure 7b). Again, Lydia’s figurative partitioning activity involved her carrying out same sequence of sensorimotor actions on her graph and circle, the elements of which entailed similar perceptual results (e.g., the sequence of drawing horizontal and vertical segments, and curves).



(a) (b)

Figure 7. (a) Lydia’s new with drawn partitions, and (b) Lydia’s circle with drawn partitions.

Discussion

Characterizing a student’s thinking of amounts of change in terms of figurative or operative partitioning activity is significant in that it allows us to describe nuances in Carlson et al. (2002)’s covariation framework and, more generally, mental actions involved in quantitative reasoning (Thompson, 2011). A student’s amounts of change understandings can differ in the extent that her partitioning activity is restricted to particular sensorimotor actions and the perceptual results of these actions. In this paper, we illustrated that a particular student’s partitioning activity was figurative because it involved her seeking to repeat sensorimotor actions in a particular order across various situations. Furthermore, her partitioning activity was constrained to having perceptually available material. Consequently, when confronted with a novel situation in which these figurative elements were absent or carrying out the sensorimotor actions failed (e.g., Which One), she had difficulty re-presenting partitioning activity.

von Glaserseld (1982) defined *concept* as “any structure that has been abstracted from the process of experiential construction as recurrently usable…must be stable enough to be re-presented in the absence of perceptual “input” (p. 194). Characterizing partitioning activity as we have enables us to extend and apply this definition in the context of students’ reasoning about relationships between covarying quantities. When a student abstracts her partitioning activity so that it is not tied to particular figurative material, thus mentally anticipating transformations of such (e.g., changing orientations or representations), she has constructed a *concept* related to this relationship (e.g., the concept of sine or rate of change). As Lydia’s activity indicates, it is important for researchers to consider students’ activities among a variety of contexts before making claims about their covariational reasoning and meanings. Moving forward, we call for continued explorations into how students reflect upon their partitioning activity and abstract quantitative relationships and structures (e.g., rate of change).

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References

Carlson, M. P., Jacobs, S., Coe, E. L., Sean, & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-379.

Desmos (Producer). (2016). Function carnival. Retrieved from https://www.desmos.com/

Ellis, A. B. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In J. Cai & E. Knuth (Eds.), *Early Algebraization* (pp. 215-238): Springer Berlin Heidelberg.

Johnson, H. L. (2015). Together yet separate: Students' associating amounts of change in quantities involved in rate of change. *Educational Studies in Mathematics*, 1-22. doi:10.1007/s10649-014-9590-y

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education, 45*(1), 102-138.

Moore, K. C., & Silverman, J. (2015). Maintaining conventions and constraining abstraction. In T. G. Bartell, K. N. Bieda, R. T. Putnam, K. Bradfield, & H. Dominguez (Eds.), *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 518-525). East Lansing, MI: Michigan State University.

Paoletti, T., Stevens, I. E., & Moore, K. C. (2017). Tricks May Inhibit Students' Reasoning. *Mathematics Teacher*(6), 446. doi:10.5951/mathteacher.110.6.0446

Piaget, J. (1976). *The grasp of consciousness: Action and concept in the young child*. Cambridge, MA: Harvard University Press.

Piaget, J. (2001). *Studies in Reflecting Abstraction*. Hove, UK: Psychology Press Ltd.

Saldanha, L. A., & Thompson, P. W. (1998). *Re-thinking covariation from a quantitative perspective: Simultaneous continuous variation.* In S. B. Berensah & W. N. Coulombe (Eds.),Proceesings of the 20th Annual Meeting of the Psychology of Mathematics Education - North America.). Raleigh, NC: North Carolina State University.

Steffe, L. P. (1991). The learning paradox: A plausible counterexample. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 26-44). New York: Springer-Verlag.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-307). Hillside, NJ: Erlbaum.

Thompson, P. W. (1985). Experience, problem solving, and learning mathematics: Considerations in developing mathematics curricula. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 189-243). Hillsdale, NJ: Erlbaum (pagination differs from published chapter).

Thompson, P. W. (2008). *Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education*. Paper presented at the Annual Meeting of the International Group for the Psychology of Mathematics Education, Morélia, Mexico.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield, & S. Belbase (Eds.), *New Perspectives And Directions For Collaborative Research In Mathematics Education., WISDOMe Monographs* (Vol. 1, pp. 33-57). Laramie, WY: University of Wyoming Press.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.

Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In L. P. Steffe, K. C. Moore, L. L. Hatfield, & S. Belbase (Eds.), *Epistemic Algebraic Students: Emerging Models of Students' Algebraic Knowing., Wisdome Monographs* (Vol. 4, pp. 1-24). Laramie, WY: University of Wyoming.

von Glaserseld, E. (1982). Subitizing: The role of figural patterns in the development of numerical concepts. *Archives de Psychologie, 50*, 191-218.