Scaling-Continuous Variation: A Productive Foundation for Calculus Reasoning

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This paper introduces a new mode of variational and covariational reasoning, called scaling-continuous reasoning. Scaling-continuous reasoning builds on Leibniz' ideas of increments and infinitesimals and does not rely on images of motion. Instead, it entails (a) imagining a variable taking on all values on the continuum at any scale, (b) understanding that there is no scale at which the continuum becomes discrete, and (c) re-scaling to any arbitrarily small increment for x and coordinating that scaling with associated values for y. We present one clarifying example of this type of reasoning and argue that scaling-continuous reasoning can support a robust understanding of foundational ideas for calculus, including rates of change, differentiation, and the definite integral.

Keywords: covariation, rate of change, infinitesimal

A curved line may be regarded as being made up of infinitely small straight-line segments.
– The Marquis de L'Hôpital, 1696

When I think of a curve, I think of a bunch of really tiny lines.
– Wesley, research participant

Introduction

Researchers argue that continuous covariational reasoning is critical for students' development of a robust understanding of function, rates of change, and the foundational ideas of calculus (e.g., Carlson et al., 2003; Kaput, 1994; Thompson & Carlson, 2017). Further, students need opportunities to reason covariationally throughout their K-12 schooling in order to be positioned to make meaningful sense of introductory calculus courses at the undergraduate level. Thompson and Carlson (2017) emphasize, in particular, the importance of smooth continuous reasoning, while also acknowledging the challenges in supporting students' ideas of smoothness. In response to these challenges, we introduce a new mode of reasoning, *scaling-continuous variation / covariation*. Building on Leibniz' notion of infinitesimal increments, scaling-continuous reasoning entails an image of the continuum as infinitely zoomable, coupled with the understanding that one can re-scale to any arbitrarily small increment for *x* and coordinate that scaling with associated values for *y*. We argue that this mode of reasoning can support productive ways of thinking about key calculus ideas, including varying and instantaneous rates of change, limit and differentiation, and the definite integral.

Background: Variation and Covariation as Foundational Ideas for Calculus

Developing a conception of quantities' values varying continuously – and consequently understanding functions as processes of covariation – is central to the emergence of calculus understanding (e.g., Carlson et al., 2003; Kaput, 1994; Rasmussen, 2000; Thompson & Carlson, 2017; Zandieh, 2000). Research suggests that students both enter and emerge from freshman calculus courses with a weak understanding of the function concept, struggling to conceptualize a function as a mapping, to use functions to model dynamic situations, and to develop robust understandings of varying and instantaneous rates of change (Breidenbach et al., 1992; Carlson, 1998; Carlson et al., 2002; Dubinsky & Harel, 1992; Monk & Nemirovsky, 1994; Thompson, 1994). One factor contributing to these difficulties is the lack of emphasis on variation in secondary mathematics; students typically do not have access to exploring functions as a way to measure variation before calculus (Cooney & Wilson, 1996; Ellis, 2011; Roschelle, Kaput, & Stroup, 2000; Thompson & Carlson, 2017; White & Mitchelmore, 1996). Thompson and Carlson (2017) argue that continuous covariational reasoning is epistemologically necessary for students to develop the foundational ideas of calculus, and moreover, students are unlikely to succeed in calculus without this foundation already in place. Students must therefore build ideas of continuous variation in secondary school in order to develop the ways of thinking necessary for meaningful calculus learning at the undergraduate level.

Researchers have addressed covariational reasoning in a variety of ways, but for the purposes of this paper we focus on work that considers the imagistic foundations that can support students' abilities to think covariationally (e.g., Castillo-Garsow, 2012; 2013; Saldanha & Thompson, 1998; Thompson, 1994; Thompson & Carlson, 2017; Thompson & Thompson, 1992). These researchers describe covariational thinking as the act of holding in mind a sustained image of two quantities' values varying simultaneously; students imagine how one quantity's value changes while imagining changes in the other. A person thinking covariationally can couple two quantities in order to form a multiplicative object (Thompson & Saldanha, 2003), subsequently tracking either quantity's value with the immediate understanding that the other quantity also has a value at every instance (Saldhanha & Thompson, 1998).

Castillo-Garsow (2012; 2013) distinguished between two types of continuous variation, which he termed chunky and smooth; Thompson and Carlson (2017) subsequently built on these distinctions to create a covariational reasoning framework. Chunky continuous variation is similar to thinking about values varying discretely, except that one has a tacit image of a continuum between successive values. This image entails intermediate values without imagining the quantity actually taking on those values (Thompson & Carlson, 2017). Instead, one imagines change occurring in completed chunks, without imagining that variation occurs within the chunk. In contrast, smooth continuous variation entails an image of a quantity changing in the present tense; one can imagine a value varying as its magnitude increases in bits while simultaneously anticipating smooth variation within each bit (Thompson & Carlson, 2017). An image of a quantity's value varying from a_1 to a_2, one will also include an image of that value passing through all intermediate measures between a_1 and a_2.

Thompson & Carlson (2017) emphasize Castillo-Garsow and colleagues' point that smooth variational thinking requires thinking about motion (Castillo-Garsow, 2012; Castillo-Garsow, Johnson, & Moore, 2013). They note that this argument "is reminiscent of Newton's description of fluents – the flowing quantities that were at the root of his calculus" (Thompson & Carlson, 2017, p. 430). Further, they point to the importance of motion to smooth covariational reasoning as well, explaining that this is akin to defining a function parametrically in terms of an underlying time variable, in which a parameter is used in the sense of a variable that is not assigned to an axis in a coordinate system. They describe the act of coordinating quantities' values as similar to forming the pair $[x(t), y(t)]$, in which the parameter "t" represents conceptual time, which is distinguished from experiential time in that it is an image of measured duration: "We are speaking of someone imagining a quantity as having different values at different moments, and envisioning that those moments happen continuously and rhythmically" (p. 445).

Smooth-continuous reasoning, which relies on this underlying image of time-parametrization, reflects one mode of robust variational and covariational reasoning. We propose an additional mode of reasoning, which we call scaling-continuous reasoning, and suggest that scaling-continuous reasoning may be both distinct from and equally robust to smooth continuous reasoning.

Scaling: An Alternative to Motion

Motion is an essential image for Newton's reasoning with variation and covariation. For Newton, a variable quantity was a "fluent," which depended on and changed with time. A "fluxion" was an instantaneous speed of this fluent's motion, and what we call a derivative is a ratio of two fluxions (Edwards, 1979). In contrast with Newton, G. W. Leibniz, the other inventor of calculus, seldom described variation, functions, and ideas of calculus in terms of motion. Instead, Leibniz attended to *differences* (differentiae) or *increments* between two values of a quantity, and he distinguished among types of these differences based on their relative *scales* or orders. For instance, Leibniz began with the notion of a "function[1]," an algebraic relationship between the values of a variable quantity such as x and the values of another variable quantity, y (Bos, 1974). Leibniz's differential calculus was then a way to derive from such a function a new equation describing the relationship between infinitesimal increments of the two quantities, $\mathrm{d}x$ and $\mathrm{d}y$. Integral calculus simply went the other way, enabling the determination of a function from a given differential equation.

Here is a brief characteristic example of Leibniz' discourse about the product rule:

> $d(xy)$ is the same as the difference between two adjacent xy, of which let one be xy, the other $(x+dx)(y+dy)$. Then $d(xy) = (x+dx)(y+dy) - xy$, or $xdy + ydx + dxdy$, and this will be equal to $xdy + ydx$ if the quantity $dxdy$ is omitted, which is infinitely small with respect to the remaining quantities, because dx and dy are supposedly infinitely small. (From Leibniz' *Elementa*, quoted in Bos, 1974, p. 16.)

There are several things to notice about Leibniz' ideas. Firstly, he began by creating a new variable quantity, xy, and then sought to derive an equation describing the *correspondence* between an infinitesimal increment of this quantity, $d(xy)$, in terms of infinitesimal increments (dx and dy) of the other two quantities x and y. These increments were static entities, although they were variable because their values depend upon where on the curve they are taken. Although Leibniz did not appeal to motion, he relied on an underlying image of every increment of one quantity, no matter how small, corresponding to an increment of an associated quantity. This is one of the crucial ideas entailed in continuous covariation (Thompson & Carlson, 2017).

Secondly, Leibniz dismissed the quantity $dxdy$ because it is infinitely small even in comparison to other infinitely small quantities such as dx and xdy. Leibniz developed a scheme of orders of the infinitesimal and the infinite in order to systematize an idea of *scaling*. For instance, at the finite scale, infinitesimals such as dy are negligible, but at the first-order infinitesimal scale they become significant, with second-order differences still negligible. The idea of imagining covariation and correspondence different scales was crucial to a coherent system of calculus for Leibniz. It is also part of successful formalizations of this system, such as nonstandard analysis (Keisler, 1986). The manner in which infinitesimal differences in the continuum become significant at different scales is illustrated by Keisler's image of a

[1] It appears that Leibniz actually coined this term. His usage differs from our modern idea in that this relationship need not be uni-valued, but must be represented algebraically.

microscope with an infinite scale factor (Figure 1).

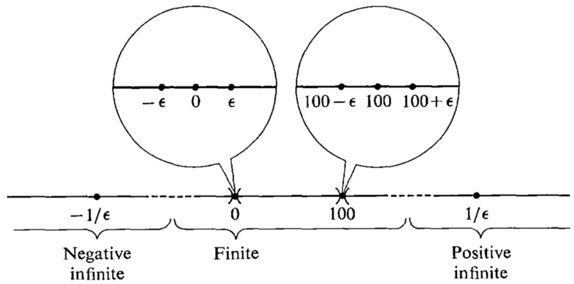

Figure 1: Infinite microscope on the continuum reveals infinitesimal increments (Keisler, 1986)

It is also notable that Leibniz' sense of variation, although not explicitly illustrated in the quote above, contained *no atomic level* of scaling. One may imagine zooming in on the continuum to smaller and smaller increments. Regardless of how far one scales, even zooming in by higher orders of infinity, at no level will the continuum ever reveal itself to be discrete points.

Finally, Leibniz emphasized "that a curvilinear figure must be considered to be the same as a polygon with infinitely many sides" (1684, p. 126). It is not that the curve *is* such a polygon, but that it can be *treated* as one. According to Bos (1974), this idea reflects the close relationship between Leibniz' idea of variation and of the infinitesimal, in which successive terms of sequences have infinitely small differences. Thus, "the conception of a variable and the conception of a sequence of infinitely close values of that variable, come to coincide" (p. 16).

Scaling-Continuous Variation and Covariation

With Leibniz' reasoning in mind, we propose a new category for variational and covariational reasoning that is distinct from the smooth-continuous category, but yet plausibly just as robust for supporting coherent and powerful reasoning with continuous quantities, functions, and rates of change. We call this category *scaling-continuous* for variational and covariational reasoning. Scaling-continuous reasoning entails the following:

1) *Variation*: Imagining that at any scale, the continuum is still a continuum and a variable takes on all values on the continuum. There is no scale at which the continuum is discrete or one reaches a point. One can conceive of the continuum as infinitely "zoomable", in which the process of zooming will never reveal any holes or atoms.
2) *Covariation*: Conceiving of a re-scale or "zoom" into any arbitrarily small increment for x and coordinating that scaling with associated values for y. One can imagine a window of x-values growing or shrinking, and the window of y-values simultaneously growing/shrinking, as a correspondence between increments of x and y.
3) This way of thinking does not fundamentally rely on an image of motion or an underlying time parameter.

Scaling-continuous reasoning reflects only one aspect of Leibniz' way of thinking about calculus. We do not treat it as necessarily entailing Leibniz' other ideas; for instance, it does not explicitly entail any particular conception about the infinite or the infinitesimal. It is possible to appeal to the image of scaling or zooming to discern or to describe covariation at different levels, without also having encapsulated an image of an infinite scale factor revealing infinitesimal increments. This latter image would require another cognitive act beyond just employing scaling-continuous reasoning. Although Bos (1974) points out that for Leibniz the idea of infinitesimal and variable are closely wed, we do not wish to presume them wed, a priori, for students who are in the process of developing covariational reasoning.

An Example: Wesley's Scaling-Continuous Reasoning

Here we introduce an episode from a teaching experiment in which the second author worked with two students who reasoned about constant and varying rates of change. One of the students, Wesley[2], communicated ideas consistent with scaling-continuous reasoning. Because this is a theoretical report, we present the following only as a motivating and clarifying example. We use as a guideline that new theoretical work should emerge through encounters with research episodes, and it was through a few key examples from students at both the secondary and undergraduate levels that we developed the theoretical category of scaling-continuous reasoning. We chose this excerpt with Wesley, a secondary student, because his descriptions of scaling-continuous ideas were both spontaneous and clearly articulated.

One aim of the teaching experiment was to introduce contexts in which students could explore situations that, to us, entailed two continuously covarying quantities. The students investigated linear, quadratic, cubic, and higher-order polynomial functions in settings emphasizing rates of change. The following episode was the result of a task in which a triangle dynamically swept out from left to right (Figure 2); students observed a movie of the sweeping action and then produced a sketch of the total accumulated area compared to the length swept.

It is important to note that placing students in situations that we as researchers conceive of as continuous does not guarantee that students will reason with those situations continuously. In fact, on the day prior, both students drew piecewise linear graphs. During this episode, however, the students produced graphs that they described as "smooth curves" (Figure 3). Wesley (W) explained why the graph should be curved, stating that on the prior day, his graph had looked piecewise linear because he had used big increments, but "if you add the tiny increments, like in between, then it curves out," indicating that straight segments were a vestige of a rough graphing process.

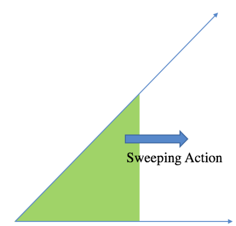

Figure 2: A static image of the triangle's area swept from left to right

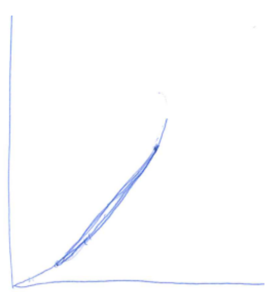

Figure 3a

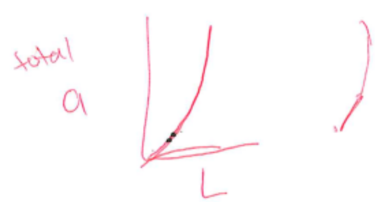

Figure 3b

Figure 3: Wesley (3a) and Olivia's (3b) sketches comparing accumulated area with length swept

To clarify, the teacher-researcher (TR) asked what the graph would look like between two points that were "super close together," marking two small black dots on Olivia's graph in Figure 3b.

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[2] Gender-preserving pseudonyms were used for all participants.

Would it be curved or straight? The following dialogue ensued:

W: I think it would, like, like these two points here (his points on Figure 3) and if you add them, do them exactly it's kind of like that and it kind of goes not straight to the curve and I think it would be more of a, a little bit more of a curve.

TR: You think it'd be a curve?

W: Yeah.

TR: And how come?

W: Because, like, there's tiny points *in between* those tiny points.

TR: Ah. There's tiny points in between those tiny points. (To Olivia): Does that make sense?

O: Yeah.

TR: What if I picked two points that were so close together that I couldn't, you couldn't even like see the difference? They were just so close together there's like an infinitesimal difference in between them. Would the connection between them be a straight line or a curve still?

O: Like the tiniest ones? [TR: Mm-hm.] Then it would be a straight line.

TR: Hmm. (Turns to Wesley). What do you think?

W: I think it'd be more of a curve because I think like it goes on infinitely kind of the points. So if you zoomed in really close on those it would like look like that and then *in between those there's still more points and it goes on forever*.

TR: Hmm. (Turns to Olivia). What do you think?

O: I still think it'd be a straight line because to me it's just a whole bunch of little straight lines and so like to me it would eventually stop because you're graphing the triangle's, like, placing, and so if you like had to choose a place to graph it from each time, then you would connect the points straight like, just straight but a whole bunch of those makes a curve, you know? (She sketches the graph on the right in Figure 3a). And so, I think this or even smaller would be the straight line. The smallest one.

TR: Hmm. What do you say to that? (Turns to Wesley).

W: I think, like, because like if they're two really tiny points right here and you zoomed in a ton it would kind of look something like these two points (in Figure 3) and then *there's still like really little points in between those points*.

The above episode reveals a contrast between Wesley and Olivia's reasoning. For Olivia, the graph is composed of straight segments. The size of those segments does not seem to be absolute, but rather to depend on a choice that is made during the graphing process. The rest of the graph is made by connecting the endpoints, but she does not treat the graph locations on the straight segments as representing quantities in the same way that the endpoints do. Here, and elsewhere in the teaching experiment, Olivia does not conceptualize variation within the straight bits of her graphs. Thus, she seems to be reasoning with, at best, chunky-continuous covariation.

In contrast, Wesley's reasoning is characterized by scaling-continuous variation and covariation. He describes variation happening within bits on his graph, and on each interval he treats the quantities' values as varying continuously, taking on all possible values within the interval. Thus, he imagines the graph to be curved on every interval, no matter how small, appealing to the idea that there are points in between the "tiny points". Wesley does not appear to rely on chunky continuous reasoning, because he fluidly rescales and imagines points in between the points, at any scale, even zooming in infinitely, explaining that this can go on forever.

However, Wesley's reasoning is not smooth-continuous variation either, because he never speaks of, nor does his imagery appear to rely on, a variable moving and tracing out values as it moves. In fact, consistent throughout the teaching experiment is Wesley's lack of reference to movement. Instead his imagery entails zooming and scaling. He explains that if you take any small increment "and you zoomed in a ton" you would see variation, "and then there's still like really little points in between those points." Furthermore, Wesley is explicit that this ability to zoom in, to rescale, "goes on forever," "goes on infinitely," and never grounds out at some atomic level. This is a crucial element of scaling-continuous reasoning, that there is continuous variation at every scale and it never becomes discrete. This entails the *recursion* Thompson and Carlson describe with smooth-continuous variation: "…the person, while reasoning variationally, is alert to the *potential need* to think about smaller intervals in precisely the same way as they are thinking about the interval that is currently in their reasoning" (2017, p. 440). This recursion extends to covariational reasoning for Wesley also; the fact that he sees the graph as curved at each new level of scaling indicates that there is covariation between the two quantities even at the new scale, and that this covariation is non-constant.

Supporting Calculus

We propose that scaling-continuous covariational reasoning may provide a robust foundation for student thinking in calculus; after all, it was instrumental in Leibniz' invention of calculus. Scaling-continuous reasoning can support an understanding of the ideas of rate of change, limit and derivative, and definite integrals, among others. For instance, Thompson & Carlson (2017) note that the idea of a function's rate of change being non-constant occurs by thinking of a function having constant rates of change over infinitesimal intervals of its argument, "but different constant rates of change over different infinitesimal intervals of the argument" (p. 452). As evidenced by Wesley's explanations, this image is a direct outcome of scaling-continuous covariation, through which one imagines zooming to an infinitesimal scale to imagine a tiny interval on which the function's rate of change is constant. Further, it can provide a foundation for developing an image of instantaneous rate of change, in which the rate of change at a point can be imagined as an average rate of change over an infinitesimal interval. This offers a natural motivation for the limit definition of the derivative.

Scaling-continuous reasoning can also support the concept of definite integral. In an undergraduate calculus course taught by the first author (Ely, 2017), students developed the idea of a definite integral as an accumulation of infinitely many infinitesimal bits of a quantity, each bit corresponding to an infinitesimal increment of the independent variable. This interpretation of definite integral, in turn, provided a robust support for meaningful modeling with integrals.

We do not suggest that smooth-continuous reasoning is unimportant for the development of key ideas about function and calculus. Indeed, we agree that it is a critical aspect of understanding the mathematics of change, including the ideas of calculus, and we support instructional efforts at all grade levels to develop conceptions of continuous covariation. Instead, we suggest that an additional form of reasoning, scaling-continuous variation / covariation, may also plausibly foster productive understandings to support learning in calculus. Given the potential for this form of reasoning to support key calculus ideas, we advocate for additional research to better understand the nature of scaling-continuous variation and covariation and its affordances for productive mathematical thinking.

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