Developing Preservice Teachers' Mathematical Knowledge for Teaching in Content Courses

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In this paper we present evidence that a) providing opportunities for PSMTs to engage with simulations of practice and b) making connections between advanced perspectives on geometry and 7-12 mathematics allows PSMTs to develop MKT in university mathematics content courses.

Keywords: Mathematical Knowledge for Teaching, Preservice Secondary Mathematics Teachers, College Geometry

Mathematics teachers draw on understandings of and connections between various knowledge bases while doing the work of teaching (Hill, Ball, & Schilling, 2008; NRC, 2010). These knowledge bases include but are not limited to typical problems for mathematics content, how that content is situated in the larger mathematical landscape, different ways students might come to know the content, and pedagogical strategies and principles that are specific to that content (Hill, Ball, et al., 2008). These types of understandings, often referred to as mathematical knowledge for teaching (MKT), should be explicitly developed in those people who seek to teach mathematics to others (Morris, Hiebert, & Spitzer, 2009; Silverman & Thompson, 2008).

Preservice secondary mathematics teachers (PSMTs) in the United States face great challenges in developing their MKT due to more demanding state mathematics standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) and a lack of opportunity to learn mathematics in ways that apply that learning to teaching situations. In response, the Mathematics Teacher Education Partnership (http://mte-partnership.org) has engaged in systematic research, development, and implementation efforts to improve secondary mathematics teacher preparation. As part of this effort, the Mathematics of Doing, Understanding, Learning and Educating for Secondary Schools (MODULE(S^2)) Project has developed, piloted, and studied the effectiveness of curricular materials in a College Geometry course. The materials, in the form of three modules, interweave aspects of MKT into a rigorous content course, can stand alone or be used to form a coherent and complete College Geometry course, and develop understanding of advanced content. In addition, the modules provide opportunities for PSMTs to develop their MKT and better understand the nature of the field of mathematics and its practice. The modules are designed to that the activities can be completed by non-education mathematics majors and education majors alike, because the work of and thinking in K-12 classrooms is a valid setting for applied problems in a mathematics course. In addition, because PSMTs will be expected to teach according to standards that view mathematics as a social construction rather than something that students receive (White-Fredette, 2010), it is critical that they learn college mathematics in an environment that both embraces this view and provides opportunity for PSMTs to develop MKT in secondary classroom contexts that also embrace this view. In this paper, we present our efforts to understand how the implementation of the MODULE(S^2) curricular materials might help PSMTs develop MKT.

Perspectives on MKT

One way researchers conceptualize MKT is to think of it as several intertwined types of subject matter knowledge and pedagogical knowledge. Hill et al. (2008) define six kinds of this knowledge in their MKT framework:

- Common Content Knowledge: knowledge to solve typical problems in a content area;
- Horizon Content Knowledge: knowledge of the larger mathematical context in which the mathematics one is teaching is situated;
- Specialized Content Knowledge: knowledge of content that is unique to or motivated by teaching situations;
- Knowledge of Content and Students: knowledge of how students think about, understand, or come to know particular mathematics content;
- Knowledge of Content and Teaching: knowledge of pedagogical strategies and principles specific to the mathematics content one is teaching; and
- Knowledge of Content and Curriculum: knowledge of available curricular resources and how to sequence instruction using those resources.

University faculty who teach mathematics content courses to PSMTs operate on two different levels with regard to MKT. First, we draw on our own MKT as we teach. Second, we can attend to how we facilitate developing PSMTs' MKT, not only in teacher preparation courses but also in mathematics content courses (Eli, Mohr-Schroeder, & Lee, 2013). This study investigates this second level. One way researchers might gain insight into *how* PSMTs' MKT develops is to analyze their responses to simulations of teaching practice assignments over a period of time. Simulations of teaching practice provide PSMTs with an opportunity to engage in enacting teaching practices by describing a realistic classroom scenario where student work or student thinking on a mathematics task is shared and PSMTs respond in some way (e.g., plan a class discussion or provide an explanation). Simulations of practice focus attention on key aspects of teaching that may be difficult for novices but are almost second nature for skilled teachers. They engage PSMTs in responding to student thinking – a valuable act in which PSMTs rarely engage (Grossman, Hammerness, & McDonald, 2009). Completing simulations of practice can help develop specific aspects of MKT, such as the ability to analyze student work to better understand the mathematical connections students make (Eli, Mohr-Schroeder, & Lee, 2013). Further, we know that teachers with stronger MKT foster student learning with greater mathematical richness and appropriateness than teachers with weaker MKT (Hill, Blunk, et al., 2008).

Silverman and Thompson (2008) provide a framework that we utilized for analyzing simulations of practice in order to determine how PSMTs' MKT developed over time. Within this framework, instruction is conceived of as the teacher creating space for students to reflect on mathematical ideas and formulate powerful understandings together in "similar and consistent" ways (Silverman & Thompson, 2008, p. 507). In this instructional setting, a teacher's MKT for teaching a particular idea can be measured by the extent to which the teacher has:

- an advanced understanding of the idea "that [carries] through an instructional sequence, that [is] foundational for learning other ideas, and that [plays] into a network of ideas that does significant work in students' reasoning" (Thompson, 2008, p. 32) – known as a *key developmental understanding* (KDU) of the idea;
- developed models of the many ways that students might come to understand the idea – known as *decentering*;
- an understanding of how others might think of the mathematical idea in a similar way;
- an understanding of the types of activities and discussions that might occur during those activities that would support others developing similar understandings of the idea;
- an understanding of how students who have come to understand the idea in this particular way are empowered to learn other related mathematical ideas (Silverman & Thompson, 2008, p. 508).

MODULE(S^2) Geometry Materials

Three geometry modules were each implemented individually as a unit within a College Geometry course during this study. In all three modules, learners were expected to be generators of knowledge while exploring geometry questions and problems. We now describe the modules.

The first module, *Axiomatic Systems*, challenges PSMTs' understandings of axiomatic systems with an opening examination of the concept of straightness in Euclidean and Spherical systems. Further explorations include discussions of other non-Euclidean axiomatic systems (e.g., projective, neutral, and hyperbolic geometries) that require PSMTs to consider how propositions and concepts defined in one axiomatic system transfer to another. Examples of activities that focus on building MKT include analyses of a classroom vignette (involving angles formed by parallel lines and a transversal) in which the teacher suggests that the class could choose a different angle relationship axiom as their starting point. This activity points to ideas about the structure of axiomatic systems and challenges PSMTs to draw on that knowledge as they consider alternative lesson structures. Another MKT development activity engages PSMTs in exploring the midpoint quadrilateral theorem and its related corollaries in Euclidean geometry. In both activities, PSMTs are required to draw on their understandings of deep, underlying concepts that form the foundation of topics taught in high school mathematics.

The second module, *Transformations*, begins with an exploration of bijective functions which map elements from the real plane to the real plane. During this investigation, learners generate definitions of transformations and isometric transformations, and it challenges their ideas of how one might explore transformations of the plane (e.g., point-by-point analysis, algebraic methods, or graphical methods). The series of activities that follow serve to deepen the PSMTs' MKT for transformations as they delve into horizon knowledge (e.g. how isometries of the plane form a cyclic group under function composition) and other related areas of pedagogical content knowledge. Other activities that seek to develop MKT in this module ask PSMTs to examine sample high school student work on constructing reflections and rotations in order to consider what the high school students may have been thinking. By the end of the module, PSMTs define congruent shapes from a transformational perspective, and they see the structure of axiomatic systems at work when they come to understand that they must introduce a reflection axiom. The module culminates with proofs of the triangle congruence theorems from a transformational perspective. Teachers with MKT rooted in understandings of geometry from a transformational perspective can help students engage meaningfully with geometric thinking instead of relying on algebraic or arithmetic methods of solving a problem (Seago et al., 2013).

The final module, *Similarity*, builds on PSMTs' understandings of transformations by adding a dilation to produce similarity transformations. Once PSMTs construct a dilation and describe it with clear mathematical language, they discover the need for having a way to measure. This propels learners into explorations of the Pythagorean Theorem and measuring area within an axiomatic system. In this module, PSMTs explore whether all parabolas are similar, and they prove the Triangle Similarity Theorems. Throughout all three modules, the materials provide opportunities for PSMTs to connect advanced perspectives in geometry to the content of K-12 geometry standards in the CCSSM. We contend that the specific efforts to include an examination of realistic classroom scenarios, sample student work, prevalent misconceptions, and connections between geometric ideas work together to develop PSMTs' MKT.

Methods

In this investigation, the second author taught a College Geometry course with 16 students that was required of PSMTs in a secondary mathematics certification program but was open to all mathematics majors. All 16 students agreed to participate in this case study designed to answer the question: How do the modules help PSMTs develop MKT? In order to measure whether or not an increase in MKT occurred during the semester, we utilized a nationally validated Geometry Assessment for Secondary Teachers (GAST) (Mohr-Schroeder, Ronau, Peters, Lee, & Bush, 2017) measure. Each PSMT took the GAST at the beginning and end of the course, and the research team scored responses after being trained by GAST staff. In order to gain insight into *how* the PSMTs' MKT changed, we analyzed pre- and post- simulations of teaching practice assignments according to the Silverman and Thompson MKT framework.

We focus on two simulations of practice in this report. In the first simulation of practice, PSMTs viewed the classroom scenario in Figure 1 and responded using the following prompts at the beginning (Pre) and end (Post) of the unit. The **Pre-assessment Prompt** was: *A class has been working on properties of quadrilaterals, specifically proving that the pair of base angles of an isosceles trapezoid are congruent. During the discussion, a student makes the statement shown in the clip. What is the student thinking? How should you respond?* The Post-assessment Prompt included two parts: ***Part 1*** *– Write the words the teacher should say in responding to this student, and* ***Part 2*** *– What do you think the student in the previous depiction meant, and what you say about the Van Hiele level of this this student's understanding?*

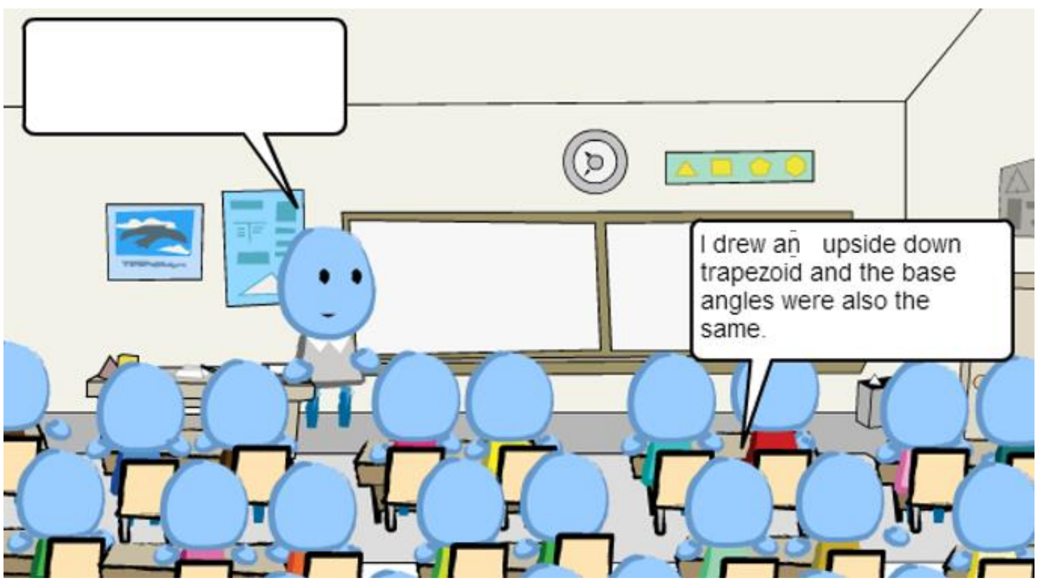


Figure 1. A teaching scenario for simulation of teaching practice assignment #1.

The second simulation of teaching practice utilized the student work in Figure 2 and the same prompt for both the pre- and post- assignment. The **Prompt** was: *Your students are working on reflection problems (reflecting segment* ***a*** *over line of reflection* ***r****). While circulating the room and observing the students' work, you encounter the two responses shown in the figures below. Explain how the students may have obtained their solutions and evaluate the result of their work. What feedback would you give the students?*

A research team of four coders analyzed PSMTs' responses to the pre- and post-assignments of both simulations of teaching practice. Here, we describe the five codes we matched to this framework. With regard to KDUs, it was difficult to find evidence to determine a) the level of advanced understanding for a PSMT's expression of a mathematical idea in the text of their response and b) whether or not the understanding was part of a network of ideas that carried through an instructional sequence. Therefore, if PSMTs made statements that were mathematically sound and we could envision the statement as being a critical piece of such a network of ideas, we coded the statement as ***KDU*** to indicate PSMTs may have at least a piece of a KDU. Because decentering means PSMTs understand models of ways students understand an idea that also differentiates their own point of view from another's point of view, we only coded for ***decentering*** when PSMTs provided more than one way of reasoning mathematically about the idea. If students showed evidence of understanding the way in which the student was thinking about the idea, we coded for ***understanding student thinking***. If the PSMTs' response suggested an activity that could be completed to advance student thinking, we coded for ***activity***. Finally, if a response showed evidence that the PSMT understands how understanding a mathematical idea in a particular way empowers the learning of another idea, we coded for ***connections***. As the team began coding, we found it important to also code for each ***incorrect mathematical statement*** as well as ***general discourse moves*** where the PSMTs sought to respond to students, promoting discourse to advance the lesson, but where the response did not specifically draw on particular mathematical ideas that would advance student thinking.

Student 1: Student 2:

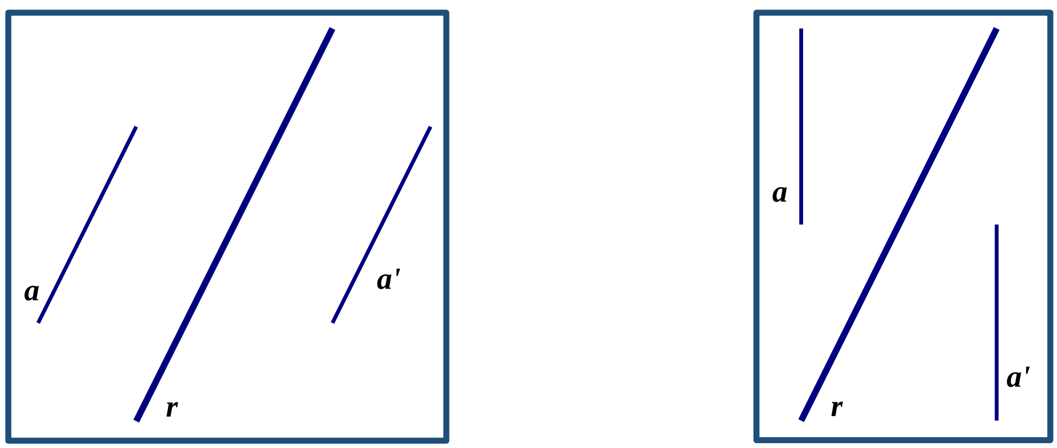


Figure 2. Sample student work used with simulation of teaching practice assignment #2.

Two coders analyzed half of the responses and a second pair of coders analyzed the other half. Then, each pair independently analyzed how the other pair coded their data, noting areas of disagreement. In this way, all four coders analyzed all of the simulation of practice data. All four coders then reassembled to negotiate disagreements in coding in order to arrive at a final coding of the PSMTs' responses. Once this step was complete, the team looked for patterns that emerged when comparing the case of each PSMT – focusing particularly on their responses to the simulations of practice pre-assignments and post-assignments.

Results

A comparison of pre- and post-GAST scores revealed that PSMTs' MKT did indeed increase. The mean score on the pre-GAST was 8.7 out of a possible 16 and the post-GAST mean score was 10.7. A paired *t*-test showed that this difference is significant with a *p*-value of 0.002. An analysis of the simulations of teaching practice showed that PSMTs provided more evidence of the presence of MKT categories on the simulation post-assignment responses when compared to the pre-assignment, particularly with regard to ***KDU***s and ***understanding student***

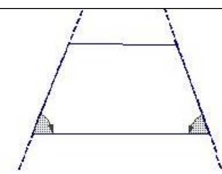
thinking. In addition, the number of instances of ***general discourse moves*** decreased dramatically from pre- to post, indicating that PSMTs progressed in their ability to respond to particular mathematical reasoning of students in their responses by the end of the course. In light of this, it is not surprising (though not necessarily encouraging) that there were slightly more ***incorrect mathematical statements*** in the post-assessments, because more PSMTs were making a greater number of mathematics-specific statements in their responses.

Here, we present responses from PSMT11, a case in the course that exhibited typical MKT development. In PSMT11's pre-response to assignment 1 (see Figure 3), we see a ***general discourse move*** asking students to state what they know or what inferences they could make about a particular idea. It is not clear how the mathematical ideas central in the question posed by PSMT11 were foundational to a connected network of ideas that could be used to advance students' understandings of base angle congruence in an isosceles trapezoid.

In contrast, part 1 of PSMT11's post-assignment response indicates ***KDU***s of rigid motions and variations in angle measure that informed the questions posed. Mathematics-specific questions that draw on understandings of rotations (upside down) and changing angle measures are clearly meant to cause cognitive conflict for the student and provide an opportunity for the student to reorganize their thinking about trapezoids. In addition, the question "Would all of our postulates and theorems still hold?" indicates an ***activity*** or class discussion that PSMT11 believes would hold potential to help the student progress in their understanding of the idea under discussion (and we agree). In both paragraphs of the part 2 response, PSMT11 provides evidence of a ***KDU*** of congruence as well as the ability to ***understand student thinking***, even if imperfectly – the student showed evidence of understanding congruency, but PSMT11 did not acknowledge it in the last sentence of the response.

Pre:

What do we already know or what inferences can we make about congruent angles in shapes?



Post:

Part 1: Would it matter if your trapezoid is 'upside-down'? Could our base angles be obtuse instead of acute? How would this affect our conclusion? Would all of our postulates and theorems still hold? Does a shape change its properties based on orientation? Why or why not?

Part 2: The student literally turned their trapezoid upside down and concluded that the 'new' base angles were also congruent. Rather than understanding that these 'new' base angles actually represented the summit angles from the right side up trapezoid, the student assumed that since the orientation of the trapezoid changed, then the base angles might also. After finding that this was not the case, the student came to the conclusion that the 'new' base angles were also congruent without deducing that the summit angles of the original trapezoid were congruent along with the base angles.

The Van Hiele level of understanding this student might possess would be considered a Level 0, which refers to Visualization. The student lacks understanding of the parts of the trapezoid functioning together, and rather views the trapezoid as a 'total entity.' This student is able to identify the trapezoid, use geometric language to describe it, and can reproduce it, but they are unaware of the special properties that it may possess, such as parallel lines and congruencies.

Figure 3. PSMT11's responses to pre- and post- simulation of practice assignment 1.

In the pre-assignment for simulation 2, (see Figure 4) PSMT11suggested an ***activity*** by posing questions about folding paper to visualize reflections and further student thinking. We also see ***activities*** in PSMT11's post-assignment, but she further elaborated by understanding student thinking and exhibiting multiple KDUs. The statement, "The student 'reflected' the original image but the student failed to notice the orientation of the line of reflection" indicates PSMT11's attempt to ***understand student thinking.*** In contrast to PSMT11's pre-assignment, ***KDU***s were prominent in the post-assignment. For example, she recognized that when an image is reflected over a line, distance is preserved and segments in the image are congruent. Additionally, PSMT 11 used her KDU of reflections to develop activities when she explained that "we must use perpendiculars to measure our distances" (***KDU***) and then posed questions that she could ask to advance student thinking (***activities***). Though we see more KDU in the post-assignment we consider this may be due to the nature of the geometry course.

Pre: If you folded your paper across the dotted line, would your reflection process still hold? Why or why not? What can you do to your conclusion so that it does hold? What does it mean for something to be reflected? Think about your answer in terms of mirroring.
Post: ***The Student's Solution***: This student noticed that the original image is parallel to the line of reflection, yet they unsuccessfully performed a proper reflection. The student 'reflected' the original image but the student failed to notice the orientation of the line of reflection. It is true that the new image is the same distance away from the line of reflection as the original and the new image is congruent to the original, but if you metaphorically 'folded your paper in half,' the image would not lie on top of itself. The new image's endpoints would not fall on the same perpendicular as the original image's endpoints, which is the main factor contributing to this misconception. The endpoints of each image must fall on the same perpendicular. The midpoint of the perpendicular will fall on the line of reflection. ***My Feedback***: What would happen if you folded your paper across the line of reflection? How does this reflection differ from your answer? Compare your answer to the actual reflection. What do you think went wrong? I know you know that the distance from the line of reflection to each image must be congruent and the lines themselves must be congruent. But think about if you were that line, and you looked directly into the line of reflection (like it was a mirror)... would that reflection be skewed to the left? Where would the reflection be? Why? This is why we must use perpendiculars to measure our distances from the line of reflection to our image. Your new image's endpoints will lie on the same perpendicular as the endpoints of your original image.

Figure 4. PSMT11's responses to pre- and post- simulation of practice assignment 2.

Discussion

In this report, we present evidence of PSMTs' development of MKT. In particular, after learning in a College Geometry course with MODULE(S^2) curricular materials, we observed the development of PSMTs' KDUs of mathematical ideas and the ability to more fully understand student thinking. In addition, PSMTs significantly decreased their use of general discourse moves. We attribute this advancement of MKT to the PSMTs completing activities that are grounded in the work of teaching. Baumert and colleagues (2010) make a similar argument, that solely focusing on common content knowledge develops "only a limited mathematical understanding of the content covered at specific levels" in the school curriculum (p.167). Providing opportunities for PSMTs to engage with simulations of practice and activities that make connections between advanced perspectives on geometry and 7-12 mathematics allows them to begin to bridge the gap between college coursework and classroom teaching and meet the challenge of developing MKT in their university mathematics content courses.

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