Students' Strategies for Setting up Differential Equations in Engineering Contexts

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Ordinary differential equations (ODEs) comprise an important tool for mathematical modelling in science and engineering. This study focuses on how students in an engineering system dynamics course organized the act of setting up ODEs for complex engineering contexts. Through the lens of ODEs as a "coordination class" concept, we examined the strategies that seemed to guide the students' interpretations of problem tasks and their activation of knowledge elements during the tasks, as the students worked to produce ODEs for those tasks. This led to our uncovering of three main strategies guiding the students' work, and the finding that being able to flexibly draw on all of these strategies may be beneficial for student success.

Keywords: differential equations, mathematics in engineering, system dynamics

Ordinary differential equations (ODEs) comprise a branch of mathematics that is extremely useful for mathematical modelling in a range of STEM (science, technology, engineering, and mathematics) fields. For example, it can be used in biology to model population dynamics, in engineering to model the evolution of mechanical system, and in physics to model changing quantities. A growing body of research has been examining how students understand, solve, and interpret ODEs in mathematics. Most of this work has focused on how students understand solution processes and the solutions themselves (Arslan, 2010; Camacho-Machín, Perdomo-Díaz, & Santos-Trigo, 2012; Habre, 2000; Rasmussen, 2001; Rasmussen & Blumenfeld, 2007). From this we know that students may struggle with the idea of a *function* being a solution (Rasmussen, 2001), that students may be hesitant about using graphical solution procedures (Camacho-Machín et al., 2012; Habre, 2000), and that equilibrium solutions are not well understood by students (Rasmussen, 2001; Zandieh & McDonald, 1999).

There is much less we know about how students organize their work for *setting up* ODEs for given contexts. Rowland and Jovanoski (2004) and Camacho-Machín and Guerrero-Ortiz (2015) each examined the setting-up process and interpretation of simple ODEs and found that students struggled to use "rate of change" thinking when doing so. They often thought of constants in ODEs as representing *constant amounts* rather than *constant rates of change*. While these studies provide useful results, the contexts used in the tasks were fairly simple and all the needed information was provided in the task. By contrast, in engineering, students encounter quite complex situations for which not all of the information is directly presented. This type of situation implies more challenges for the students as they attempt to organize their work to produce an ODE. We believe it important to extend the research on ODEs by examining how students go about the process of setting up differential equations for tasks that involve complicated systems. In summary, this report is meant to investigate the research question: What strategies do students use when setting up ODEs for complex engineering tasks?

Coordination Class Concepts

For this study, we used the lens of coordination classes from the knowledge-in-pieces paradigm (diSessa & Sherin, 1998). Coordination classes are useful for describing concepts whose purpose is "getting information" (p. 1171). In the context of system dynamics, the information regarding the system is obtained through an ODE. A coordination class concept involves readout strategies and causal nets. *Readout strategies* are the "means of seeing things that relate to the target information" (diSessa, 2004, p. 141). For our purposes, we see readout strategies in terms of how one interprets external stimuli. The *causal net* is "the set of all possible inferences that lead to determining the relevant information" (diSessa, 2004, p. 141). That is, once a person has interpreted external things, those interpretations can then be linked with other pieces of knowledge so as to progress toward the desired information. Causal net elements might consist of known relationships, formulas, informal ideas, beliefs, and so on.

Next, diSessa and Wagner (2005) define a *concept projection* as the collection of knowledge elements related to that concept, as well as the guiding strategies, that a person uses in a particular context. The strategy used to obtain the information impacts the readouts and causal net elements that are activated. For example, suppose one wants to find the volume of a three-dimensional object (the desired information). One might first use a readout strategy to identify whether the object's shape is a typical geometric shape, or not. If it is, such as a box, the person might activate the causal net knowledge element $V=lwh$. Using this geometric formula, they could obtain the object's volume. We could call this a "geometric strategy," because the person used the geometric regularity of the object to determine how to find the information. On the other hand, if the object is an irregular shape, the person might instead activate a causal net knowledge element of Archimedes principle, which states that the volume displaced by water is equal to the volume of the submerged object. The person could then use that inference to determine the object's volume. We could call this a "experiment strategy," because the person would enact an experiment based on a known principle to determine the information.

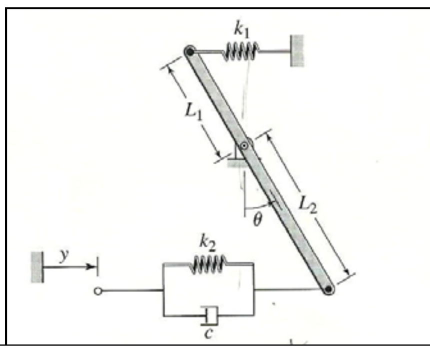
Data Collection and Analysis

In order to provide insight into the strategies students use to set up ODEs for complex tasks, we recruited students for interviews who were taking an engineering "system dynamics" course. We chose a system dynamics course because (1) taking an ODE course is a prerequisite for the system dynamics course, meaning all the students had experience with ODEs; and (2) the system dynamics course is designed entirely around the idea of setting up and solving ODEs for different engineering systems. Thus, the students were in the process of learning to set up ODEs for complex contexts, though the instructor mostly just lectured on how to set up ODEs. To recruit students from the system dynamics class, we first administered a survey to the entire class to obtain background information on how the students interpreted a generic ODE. The survey displayed the equation $ay''+by'+cy = 0$ and asked the students to describe what this equation meant and what the various symbols in it represented. We chose two students who provided strong responses regarding the equation (Rebecca and Zane), two students who provided moderate responses about the equation (Harry and Josh), and one student who showed some weaknesses in their understanding of the equation (Kira). These five students participated in two interview sessions where they were asked to set up an ODE for a total of three different tasks.

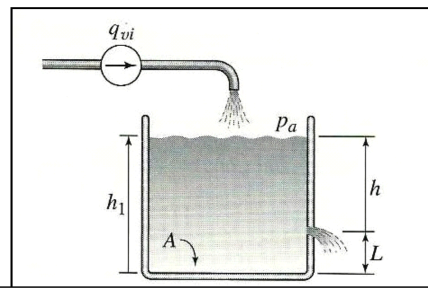
We designed the interviews to focus on contexts that matched those seen in the students' system dynamics course. The three tasks consisted of a mechanical context, an electrical context, and a fluid context. For the purposes of this abbreviated conference report, we focus on the mechanical task (task 1) and the fluid task (task 3), shown in Figure 1, as they suffice for describing the main strategies the students used for organizing their work of setting up an ODE. For the interview, the students worked out the tasks, explaining their thinking aloud, and the interviewer asked follow-up and clarifying questions while the student worked.

The interview data were analyzed in two separate phases. In the first phase, which was essentially a preliminary phase in terms of this paper's research question, we identified readouts and causal net elements the students used while working on the tasks. Operationally, readouts were defined as any place in the data where a student appeared to make a direct interpretation of any part of the given interview task, whether symbols, words, or parts of the figure. The apparent interpretation was recorded as the "readout." Causal net elements were operationally defined as any time a student mentioned, wrote, or suggested an idea that was not a direct interpretation of a part of the task. The substance of the causal net element, as well as what other piece(s) of information may have triggered its activation, was recorded as a "causal net link."

In the second phase, which allowed a more direct answer to this paper's research question, we used the resulting readouts and causal net links recorded in phase one to examine the overall flow of the students' work. This allowed us to infer strategies the students appeared to be using to set up the ODE. We did not have pre-set notions of what the strategies would consist of, but rather let the nature of the strategies emerge from the student's documented process. This led to the identification of three main strategies, described in the next section. Lastly, we determined whether the strategies were productive for the students, by observing (1) whether a particular strategy helped the students produce a solution, (2) whether that solution was correct, and (3) whether the student had to revise their solution because the approach led to a "dead end."



Task 1: Assuming that θ is small, derive the equation of motion of the pendulum shown in the figure. The input is $y(t)$ and the output is θ. The equilibrium corresponds to $y = \theta = 0$.



Task 3: The cylindrical tank shown in the figure has a circular bottom of area A. The volume inflow rate from the flow source is $q_{vi}(t)$, a given function of time. The orifice in the side wall has an area A_0 and discharges to atmospheric pressure p_a. Develop a model of the liquid height h.

Figure 1. The mechanical and fluid tasks from the interviews (taken from Palm, 2005, p. 244 and 397).

Results

In this section, we describe, one by one, the three main strategies used by these five students to guide their work setting up ODEs for the tasks. We do this by providing a single illustrative case from the data for each of the three main strategies. We then provide a summary about which students used each strategy, and end by discussing possible benefits of the three strategies.

Strategy #1: Diagram-based Approach

To illustrate the first strategy, we describe the case of Zane working on task 1. While his complete work is too lengthy to describe here in its entirety, we highlight enough of his work to hopefully demonstrate his main guiding strategy. An important early readout in Zane's work was to view the bar in the figure as the main object about which to reason. This led Zane to draw what, in engineering, is called a free-body diagram. His diagram consisted of the bar, by itself, which he continually annotated and revisited throughout his work (see Figure 2). The diagram helped him focus on two other readouts, namely the top of the bar and the bottom of the bar (arrows at the top and bottom of his diagram). He then inferred that forces and horizontal displacement were both relevant attributes of the top and bottom of the bar. The top arrow was linked to a single force from the upper spring, while the bottom arrow was linked to two forces from the lower spring and the damper. His diagram also helped him visualize the implicit presence of two right triangles each having θ as an angle (added to Figure 2). He used the triangles, together with Hooke's Law ($F=k\cdot\Delta x$) and the fact that $\sin(\theta) \approx \theta$ (inferred from the "small angle"), to describe the three forces in terms of L_1, L_2, and θ (seen in Figure 2 as $k_1L_1\theta$ and $k_2L_2\theta$ for the springs, and $cL_2\dot{\theta}$ for the damper). Causal net links between *velocity* and *first derivative* and *rotational acceleration* and *second derivative* allowed him to invoke the $\dot{y}$, $\dot{\theta}$, and $\ddot{\theta}$ seen around the "cloud" in Figure 2. Zane used the standard engineering "dot" notation for time-derivatives, as did the other students. While there are additional science-based knowledge elements Zane used along the way, such as a "moment" being force times the distance from the center of rotation, and $\sum moments = I\ddot{\theta}$, we can see that most of his readouts and causal net links were scaffolded by his free-body diagram. Putting all of these elements together, Zane was successful at producing a correctly set up ODE for this task, shown in Figure 3.

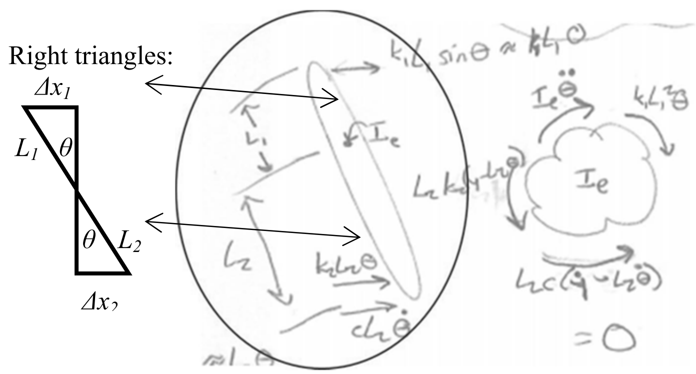


Figure 2. Zane's initial free-body diagram (circled) and his work based off the diagram

$$I_e\ddot{\theta} + k_1L_1^2\theta - L_2c(\dot{y} - L_2\dot{\theta}) - L_2k_2(y - L_2\theta) = 0$$

Figure 3. Zane's correctly set up ODE for task 1.

Having briefly recapped Zane's work, we can see that his overall strategy consisted of using a single diagram to organize most of his readouts and causal net links. Thus, we call Zane's strategy the "diagram-based" approach. In general, we can think of the diagram approach as a *general-to-specific* method that initially focuses on the entire system. Then, within that system, the student can attend to individual parts that have relevance to the system. The diagram approach is not limited to mechanical free-body diagrams, but can also be seen in "schematics" for electrical contexts and "control volumes" for fluid contexts. In fact, the simple existence of names for these types of diagrams in various engineering contexts suggests its generalizable usefulness as a strategy for setting up equations, which apparently extends to ODEs as well.

Strategy #2: Components-based Approach

The second strategy we describe contrasts with the diagram-based approach in that it could be considered a *specific-to-general* strategy. To illustrate it, we describe the case of Rebecca also working on task 1. Unlike Zane, who first "read" the bar in isolation, Rebecca's initial readouts were to scan the task to locate and identify various individual elements and to begin to keep track of them. She immediately identified three elements, k_1, k_2, and c, each as representing forces. This is different from Zane, who initially began with only two elements, namely the top and bottom of the bar. Thus, we can see a distinction in what these strategies might focus on. She then made the causal net link that each force multiplied to its distance from the center of rotation gives the moment at that point. Using these individual components, and the fact that the sum of moments equals $I\ddot{\theta}$, she wrote an early version of the ODE, shown in Figure 4.

$$F_{k_1} L_1 + F_{k_2} L_2 + F_c L_2 + I\ddot{\theta} = 0$$

Figure 4. Rebecca's initial equation, focused on compiling individual elements of the system.

Rebecca then returned to each individual element in order to flesh each one out, which resembled Zane's work at this point. She similarly inferred $\sin(\theta) \approx \theta$ from the "small angle" in order to elaborate on F_{k1}, F_{k2}, and F_c. This approach is visible in her work shown in Figure 5, where she used a string of causal net links to establish how each element was related to y, $\dot{y}$, θ, $\dot{\theta}$, and $\ddot{\theta}$. After finding each element in Figure 4 in terms of these variables, she combined them into a correctly set up ODE, shown in Figure 6.

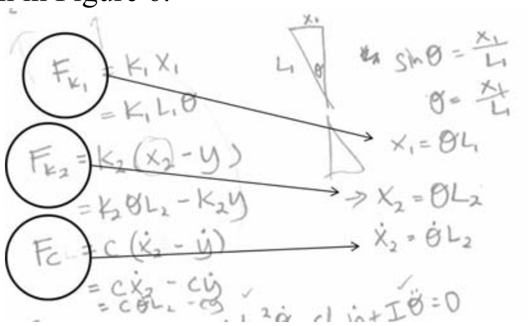


Figure 5. Rebecca's work of focusing on each element and how it could be represented in terms of θ.

$$I\ddot{\theta} + cL_2^2\dot{\theta} + k_1L_1^2\theta + k_2L_2^2\theta = cL_2\dot{y} + k_2L_2 y$$

Figure 6. Rebecca's correctly set up ODE for task 1.

In Rebecca's work, rather than beginning with a diagram, we can see the strategy of reading out specific elements *first* and *then* subsequently trying to piece them together. Of course, there were many overlapping readouts and causal net inferences with Zane's work, once she performed her initial organization of the task. Also, it is certainly true that Rebecca *did* employ holistic thinking in her work, evidenced by when she put the various components together, like in Figure 4. However, what is different and noteworthy is that her guiding initial strategy was "reading" the task through the identification of each individual element and then figuring out how to compile them. For Rebecca, the individual elements seem to have come first, and then knowledge pieces were used to organize the elements into a coherent whole. We can see that, for Rebecca, this strategy was just as successful as Zane's diagram approach, since it provided a direct path toward creating a correct ODE for this context.

Strategy #3: Equation-based Approach

For the third strategy, we again describe Rebecca's work, but this time with task 3. An important initial readout for Rebecca in task 3 was simply to attend to the general *fluid flow* context of the problem, as opposed to any individual element within it. Her recognition of this type of context seemed to immediately activate a causal net link that an adaptation of Bernoulli's equation governs these types of fluid contexts. This link allowed Rebecca to immediately invoke an entire equation as a single knowledge resource, $q_{in} - q_{out} = \rho\dot{V} + \dot{\rho}V$ (where q is a flow rate, ρ is the fluid's density, and V is the fluid's volume in the container). That is, rather than *piecing together* an equation, as Zane and Rebecca (and other students) did for task 1, in this case an entire equation was *recalled* from memory because of its relevance to the context. The remaining work for Rebecca in this task was then to manipulate this equation by making substitutions or cancelations that would produce the desired ODE.

To do so, Rebecca first used the readout of "water" to infer incompressibility, meaning that the density would not change and $\dot{\rho} = 0$ (see Figure 7, and note the scribbled out "$\dot{\rho}V$" above the last term). Next, she used the facts that $V = Ah$, and that the cross-sectional area was constant, to substitute $\rho A\dot{h}$ in place of $\rho\dot{V}$. Lastly, she used Toricelli's Law, $q_{out} = c\sqrt{2gh}$, to make a substitution for q_{out} (where h is the distance between the fluid surface and the outflow). Notice that in her final equation (Figure 8), she could not recall exactly what was supposed to be "inside" the square root, and so her expression diverges a little from a "correct" solution. However, had she had access to a book or sheet of equations, she could have easily corrected this and thus we still consider her final ODE to essentially be "correct." Also, for clarification, her "sgn" term is the "sign function" for whether the argument is positive or negative.

$$q_{mi} - q_{mo} = \rho\dot{V} + \frac{\rho_o V_o}{\beta}\cancelto{0}{\dot{\rho}}$$

Figure 7. Rebecca's direct invocation of an entire equation governing fluid flow.

$$\rho q_{vi} - k_1\sqrt{|\rho g h - P_a|}\,\mathrm{sgn}(\rho g h - P_a) = \rho A\dot{h}$$

Figure 8. Rebecca's essentially "correct" final ODE for task 3.

In general, Rebecca's equation approach seemed to rely on the fact that there was a single main equation governing that particular class of systems. From that equation, Rebecca centered all her efforts to obtain the ODE by manipulating the equation through substitutions or cancellations. Thus, the use of this strategy would first require the perception (i.e. causal net link) that there is, in fact, such an equation that can be used for a given system. This strategy was also successful in that it provided Rebecca a clear path toward an essentially correct ODE.

Approaches Used by All of the Students

We now provide a brief summary of all five students in terms of which strategies they used (diagram, component, or equation) and whether they were successful, partially successful, or unsuccessful at setting up an ODE (see Table 1). We note that we allowed "successful" set ups to include equations where there was a simple recall mistake, like Rebecca's in task 3. We considered "partially successful" set ups to be those that had one or two significant flaws (beyond simple recall) but that still contained many correct elements in the equation. An "unsuccessful" set-up was one where the student never produced a final equation, or one in which the equation had multiple major flaws.

We see in Table 1 that not all students confined themselves to a single strategy for a given task. Harry, Josh, and Kira each used multiple strategies for at least one task in order to help them progress in their work. In some ways, because these students struggled more than Zane and Rebecca, who each only used one strategy per task, one might conclude that using more strategies is a sign of weakness. However, we do note that using different strategies actually allowed Harry, Josh, and Kira to each make more progress than they would have otherwise made with a single strategy alone. That is, once they were stuck, switching modes to a different strategy often seemed to unlock additional causal net links that may have been hidden from them while using the other strategy, even if they did not fully reach a completed ODE.

Table 1. Strategies used and whether the student was successful (S), partially successful (PS), or unsuccessful (UN)

	Rebecca	Zane	Harry	Josh	Kira
Task 1	Component (S)	Diagram (S)	Component (PS)	Diag/Comp (PS)	Diag/Comp/Eq (UN)
Task 3	Equation (S)	Equation (S)	Diag/Eq (S)	Diag/Eq (PS)	Diag/Eq (S)

Discussion of the Three Strategies

We believe the empirical documentation of these three strategies provides some insight into how students might identify and use information relevant to ODEs in complex contexts through readouts and causal nets. While experts might see the strategies as equivalent, we believe there are nuances to each, and this report may be seen as an "unpacking" of possible ways to reason about ODEs for complex engineering tasks. In fact, our study suggests these strategies could be important in *developing* expertise. All three approaches were used by students to correctly set up ODEs for these complex tasks, or at least to construct partially correct ODEs, as seen in Table 1.

We can see that there is not necessarily "one correct strategy" for a given problem. Rebecca and Harry both used the component approach to productive ends for task 1, but Zane and Josh both used the diagram approach instead to make progress on that task. While all of the students used the equation approach on task 3, Harry, Josh, and Kira also used the diagram approach (in the form of a "control volume") to help further their work. Yet, while there may not be one correct strategy, we observe that the trends in Table 1 suggest some strategies being more easily invoked for some tasks than others. The equation approach was hardly used at all for task 1, but was used extensively for task 3. This gives evidence that some problems may lend themselves better to bringing in an overarching governing equation. For example, task 1 could be considered to have the governing equation $\sum moments = I\ddot{\theta}$, but this is not where these students tended to start. Rather, this equation emerged as a causal net link further down the line, once the students were ready to organize the elements into a whole. By contrast, the fluid flow equation seemed readily available as an immediate starting place for task 3. Thus, developing expertise in setting up ODEs in these types of contexts may have something to do with being able to recognize when it may be best to start with a diagram, start with individual components, or start with an equation.

Since using multiple strategies helped the weaker students in this study make more progress than they otherwise would have with only a single strategy, evolving expertise could also partially deal with being able to switch strategy modes if a roadblock is reached within a task. Perhaps it is true that greater expertise may lead to better identification of a single productive approach, as with Rebecca and Zane. However, as students mature *toward* that point, becoming aware of which of the three strategies they are using may help them see the value in switching between strategies for a given task. This may help them develop better flexibility in which strategies they use, and to begin to see connections between certain problem types and certain strategies that are useful for that type. We see this exploratory study as a useful step, in that it could be expanded into a teaching experiment to confirm, refute, or nuance these results.

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