Investigating Students' Meta-Level Object-Reflections and Discourse-Reflections: The Provocative Power of Primary Historical Sources

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We discuss research based on Sfard's theory of mathematics as a discourse, which we used to investigate the potential of engagement with primary historical sources for motivating undergraduate students to participate in and accept new mathematical discourses. This preliminary report focuses on characterizing the nature of students' participation in mathematical discourse in their written work on primary source projects (PSPs), as well as the question of what constitutes evidence of students' noticing of meta-level rules in that work. We present our analysis of a brief excerpt from one PSP, and provide an analysis of two student work samples to exhibit students' object- and discourse-reflections at the meta-level.

Keywords: Primary Historical Sources, Analysis, Rigor, Metadiscursive Rules

Introduction

Recently we initiated a study based on Anna Sfard's theory of mathematics as a discourse to investigate the potential of engagement with primary historical sources for motivating students at the undergraduate level to participate in and accept new mathematical discourses. Part of a larger project focused on the use of primary sources in the teaching and learning of undergraduate mathematics, the investigation we report on here seeks in particular to contribute to the growing body of research on the "metadiscursive rules" that govern participation in a mathematical discourse community (Sfard, 2008).

Prior research suggests, for instance, that engagement with primary historical sources may help students learn the metarules that govern mathematicians' discourse (Kjeldsen & Blomhøj, 2012). Given these and related findings, we believe it is important to look more closely at students' interactions with unfamiliar mathematical discourses and investigate their progress in "figuring out" (Sfard, 2014, p. 201) the meta-level rules that govern a new mathematical discourse as a result of those interactions. We are further interested in determining the extent to which students' (verbal/written/other) actions both during and after engagement with the primary source projects provide evidence of their acceptance of a new discourse. In this preliminary report, we focus on the following two questions within the context of an undergraduate analysis course:

- How can we characterize the nature of students' participation in mathematical discourse in their written work related to primary source projects?
- What constitutes evidence of students' noticing of meta-level rules in this written work?

Theoretical Framing and Literature

In an attempt to resolve certain quandaries related to mathematical thinking and learning, Sfard (2008) operationally defined thinking as a personalized version of communication. Given the collective nature of communication, she introduced the term *commognition* to highlight the communicative nature of activities in our minds, emphasizing that individual cognitive processes (thinking) and interpersonal communication are "but different manifestations of basically the same phenomenon" (Sfard, 2008, p. 83). Using this communicative, or discursive lens, Sfard

(2008) determined that "mathematics begins where the tangible real-life objects end and where reflection on our own discourse about these objects begin" (p. 129). Cobb, Bouf'i, McClain, and Whitenack (1997) also noted the connection between students' mathematical development and *mathematizing* (or *reflective*) *discourse*, which they described as "characterized by repeated shifts such that what the students and teacher do in action subsequently becomes an explicit object of discussion" (p. 258). That is, what identifies the objects of communication in mathematics is their discursive nature: they come to exist as we talk about them. From this viewpoint, mathematics emerges as a highly situated human activity which generates itself. As a result, the learner of mathematics faces a paradoxical situation: How can a person join a discourse for which familiarity with the discourse is a precondition for participation in that discourse?

As a further complication, Sfard (2008) noted that participation in any discourse requires adopting the rules that govern that discourse, in addition to becoming familiar with the objects of the discourse. She referred to the former rules as *meta-level*, or *metadiscursive*, and the latter as *object-level*. For instance, asserting that a particular function is differentiable constitutes an object-level narrative about functions. However, a student's method of justifying this assertion (e.g., sketching a graph versus an $\varepsilon-\delta$ proof) would be indicative of the metadiscursive rules that govern her discourse about functions. Despite the usual implications of the word *rule* as being invariable and strictly deterministic, metadiscursive rules are subject to change in time and space, and are tacit, contingent, constraining, flexible, and value-laden. Sfard posits that these characteristics render meta-level learning possible only through direct encounters with a new discourse that is governed by meta-level rules different from those governing the learner's current discourse (p. 256). Furthermore, such encounters generally entail a *commognitive conflict* when the discursants unknowingly operate under completely different meta-level rules.

Given their role in governing the actions of the participants in a mathematical discourse, researchers have paid particular attention to factors that affect the learning of metadiscursive rules in mathematics. In a number of these studies, the history of mathematics, and primary source readings in particular, emerged as an instructional approach with strong potential to promote such learning. In their study of university mathematics students, for example, Kjeldsen and Blomhøj (2012) showed that a careful selection of historical sources can help students learn about the metadiscursive rules that govern mathematicians' discourse about functions, and allow them to recognize that these rules changed during the development of that concept. This meta-level learning, they argued, fostered students' learning of mathematics at the object-level as well. In her teaching experiment with pre- and in-service teachers, Güçler (2016) designed an instructional sequence in which the metadiscursive rules implicit in various historical sources were made explicit to students. She showed that by reflecting on their own and mathematicians' definitions of function, students experienced changes in their discourse; within the commognitive framework, it is precisely such changes that constitute evidence of learning.

Data Sources and Methods of Analysis

For our metadiscursive rules investigation, we collected data in a one-semester *Introduction to Analysis* course for senior mathematics majors. The instructor (the second author) has extensive experience in the development and use of primary source materials for teaching undergraduate mathematics courses. During the semester in question, students completed two Primary Source Projects (PSPs). Analysis PSP #1 (Barnett, 2017a) examines nineteenth century concerns about the foundations of analysis that led to an increase in formal rigor at that time; it was implemented in the second week of class through a combination of individual advance

reading/preparation followed by 1.5 days of whole-class discussion. Analysis PSP #2 (Barnett, 2017b) also relates to standards of rigor in analysis, but within the context of counterexamples satisfying certain function properties (e.g., a continuous but nowhere differentiable function). This PSP was implemented over two weeks via a combination of individual advance reading/preparation, whole class discussion, and small group work. A traditional textbook (Abbott, 2015) was also used in the course. Students were guided in their reading and study of the PSPs and the textbook by daily "Reading and Study Guides" (RSGs) prepared by the instructor.

The data collected for this study include video recordings of all class meetings, audio recording of each group during small group work for Analysis PSP #2, students' written work on both PSPs and the related RSGs, instructor class notes, pre-interviews with nine students prior to work on Analysis PSP #2, and post-PSP interviews with two of those nine. We also implemented four student surveys: a pre-course survey, two post-PSP surveys, and a post-course survey.

Since our goal in this report is to share our preliminary findings related to the evidential foundation of this metadiscursive rules investigation, we limit our analysis to just one data source: students' individual written work on Analysis PSP #2 and related RSGs. One reason for this choice is that written work is generally narrower in terms of the variables involved. In particular, students' written work allows us to focus exclusively on the individual's interactions with the material, in contrast to interview or small group work data that also involves students' discourse with each other, the instructor, and/or the interviewer. However, the primary source excerpts and the student tasks contained in Analysis PSP #2 do include considerable breadth and variety of discourse. We thus anticipate that the preliminary analysis and findings we present here will serve as a useful guide to our analysis of the more complex data sources which we will need to complete in order to align our investigation with Sfard's situated-learning framework.

For this preliminary report, we analyzed the PSP itself, the related RSGs, and students' written work on these instructional materials. Given our focus on metadiscursive rules, the main consideration that guided our analysis was whether and how the written narratives of the different discursants did or could provide indications of the implicit rules governing the various discourses. In our analysis of the PSP, for example, our goal was to identify its potential to motivate student noticing of and/or reflection on the various metadiscursive rules, either those of the student or of the discursants in Analysis PSP #2. In that PSP, there are three different discursants: Darboux, Houël, and the project author. Darboux's and Houël's voices are represented through excerpts drawn from letters exchanged during a ten-year correspondence in which they debated issues related to rigor in analysis. The voice of the project's author is present in the background narrative that describes the historical context, in the selection of particular excerpts from the Darboux-Houël correspondence, and in the student tasks based on those excerpts. The instructor-prepared RSGs directed students to read specified portions of the PSP and complete preliminary work on certain PSP tasks for the next class period.

In our analysis of student work on the RSGs, we were interested in aspects of students' written work that could be interpreted as "talking" about the actions of the PSP discursants. We completed this analysis in three stages. First, we examined student responses on specific PSP or RSG items that our document analysis identified as having potential to provoke a meta-level response, and made note of those responses in which students wrote about the actions of the discursants. In the next stage, we examined each of those student responses in detail. As we completed this analysis, we became aware that students' meta-level responses could be further classified as either reflections about specific mathematical objects, or reflections about the

discourse itself. Finally, based on this new sub-categorization scheme, we analyzed student written work on all PSP and RSG items to determine if we could document evidence for students' noticing of metadiscursive rules in the form of meta-level reflections of either kind.

Findings

Based on our analysis, we developed a two-tiered categorization scheme for student discourse in their written work. First, students produced narratives at either the object-level (i.e., they simply "did the math"), or at the meta-level (i.e., they "talked about" – or reflected on – doing mathematics). Second, the focus of students' meta-level reflections was either on the mathematical discourse in the PSP, or on the mathematical objects under discussion in that discourse. We limit our attention in this preliminary report to the second tier of this categorization scheme and consider the focus of students' meta-level reflections, as we believe these findings best characterize the nature of students' noticing of metadiscursive rules. For each subcategory within this tier, we also analyze one student response for evidence of such noticing.

We start with a short sample from the PSP to illustrate how its design could motivate students' noticing of metadiscursive rules. In Figure 1, we read from two of the discursants in the PSP: the author in the narrative before the excerpt, and Darboux in the excerpt itself. As noted by the PSP author, Darboux had been raising his concerns regarding the rigor in Houël's proofs for a fairly long time. We argue that this disagreement results from a difference in the metadiscursive rules that govern their respective discourses regarding rigor. As Darboux noted in his letter, there was a shift occurring in the discourse on rigor among mathematicians of that time, which he felt implied that "no one would find [Houël's reasoning] rigorous." We believe that, through their observations of these shifts in the discourse, students will come to notice the metadiscursive rules that govern the discourse, and begin to experience the commognitive conflict required for a shift in their own metadiscursive rules as a result.

In this same letter, Darboux also returned to a second specific concern that he had about the proofs that Houël's provided for certain theorems involving derivatives. In fact, Darboux raised this other concern as early as January 18, 1875, when he wrote[2]:

Here is what I reproach in your reasoning which no one would now find rigorous. When we have

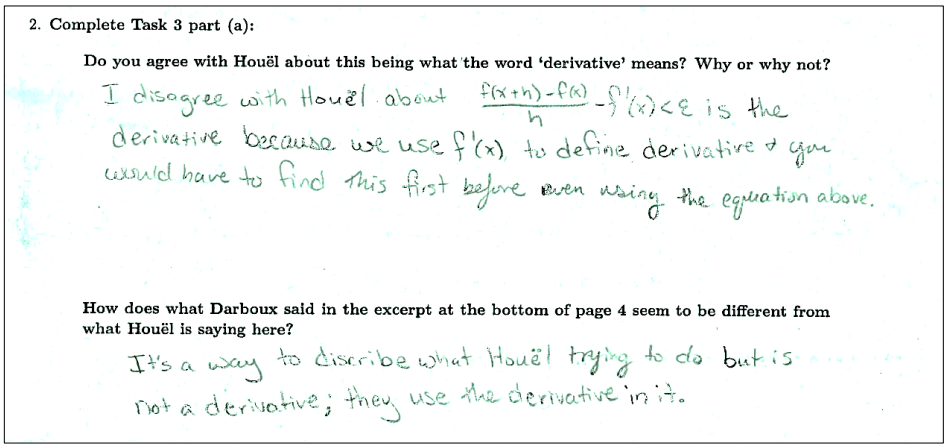
$$\frac{f(x+h)-f(x)}{h} - f'(x) = \epsilon,$$

ϵ is a function of two variables x and h that approaches zero when, x remaining fixed, h approaches zero. But if x and h [both] vary as they do in your proof, or worse yet, if to each new subdivision of the intervals $x_1 - x_0$ there arise new quantities ϵ, then I find it altogether unclear and your proof has nothing but the appearance of rigor. [Darboux, as quoted in (Gispert, 1983, p. 99)]

Figure 1. Excerpt from Analysis PSP #2 (Barnett, 2017b).

We now share a representative student response related to this particular excerpt, and a subsequent statement made by Houël about the inequality $\frac{f(x+h)-f(x)}{h} - f'(x) < \varepsilon$ and the meaning of the word "derivative." Figure 2 displays a student response that we characterized as an *object-reflection*: although the RSG prompt invites participation at the meta-level, and the student is engaged with the discourse of the PSP, her response focused on talking about the

mathematical object "derivative," rather than about the discourse itself. Analyzing this response for the student's noticing of metadiscursive rules in the discourse, we highlight her response to the second part of the RSG item: "It's a way to discribe *[sic]* what Houël *[sic]* trying to do but is not a derivative; they use the derivative in it." We are aware of a potential objection here, in that the student provided neither an explicit or implicit narrative on the metadiscourse. However, we interpret the commognitive conflict she appears to have experienced regarding definitions in mathematics – that terms should not be explicitly used in the equations/inequalities that define them – to be a result of her noticing of the disagreement between the metadiscursive rules that governed Darboux and Houël's discourses.


2. Complete Task 3 part (a):

Do you agree with Houël about this being what the word 'derivative' means? Why or why not?

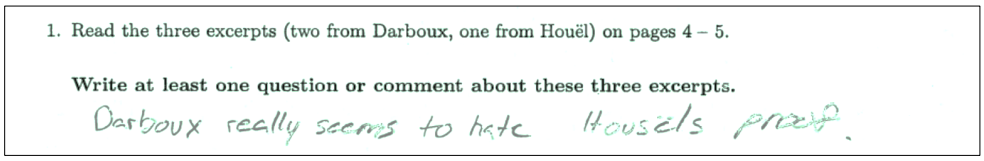
I disagree with Houël about $\frac{f(x+h)-f(x)}{h} - f'(x) < \varepsilon$ is the derivative because we use $f'(x)$ to define derivative & you would have to find this first before even using the equation above.

How does what Darboux said in the excerpt at the bottom of page 4 seem to be different from what Houël is saying here?

It's a way to discribe what Houël trying to do but is not a derivative; they use the derivative in it.

Figure 2. Student object-reflection at the meta-level (Student Response, RSG).

In the sample of a *discourse-reflection* student response shown in Figure 3, we argue that the student noticed the tension that arose between Darboux and Houël surrounding their lack of communication: by talking about (the nature of) the discourse itself, the student participated in the discourse at the meta-level. We also pay attention here to what the student did not say, as well as what he said. The student did not, for instance, evaluate Darboux's or Houël's letters for mathematical correctness, but provided instead a statement regarding the nature of their communication that has a metadiscursive characteristic. Again, we acknowledge the potential criticism that the student did not explicitly talk about the rules that might be governing Darboux's and Houël's discourse; he did, however, clearly notice the ineffective communication between them, which, in time, created the tension in the letters.


1. Read the three excerpts (two from Darboux, one from Houël) on pages 4 – 5.

Write at least one question or comment about these three excerpts.

Darboux really seems to hate Houël's proof.

Figure 3. Student discourse-reflection at the meta-level (Student Response, RSG).

Although space considerations allow us to share only very brief examples to suggest the richness of our data set, we anticipate that analysis of further examples will open up discussion of other topics of research interest, including the role of commognitive conflict in promoting metalevel learning and the implications of the discursive framework and our classification scheme for instructional practice.

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