The Relationship Between Students' Covariational Reasoning When Constructing and When Interpreting Graphs

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Abstract: Graphing tasks require students to engage in at least one of two activities: construct a graph and/or interpret a graph. Ideally, the meanings a student re-presents when constructing a graph are consistent with the meanings the student constructs from his/her sketched graph. However, this coherence is nontrivial. In this paper I present results from clinical interviews with university precalculus students to illustrate how students' graphing actions can be governed by different images of covarying quantities. More specifically, I present two students' mathematical activity to illustrate how these students' imagined quantities to covary in different ways depending on whether they were reasoning about a situation, constructing a graph, or reasoning about that sketched graph. I conclude by hypothesizing that the way a student coordinates two quantities' measures (e.g., asynchronous coordination of varying quantities or static coordination of measures) can inhibit him/her from imagining the same covariational relationship when constructing and interpreting graphs.

Keywords: Graphing, Covariational Reasoning, Cognition

Researchers have found that students who imagine quantities to covary in a situation are not necessarily able to re-present that imagery graphically (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Moore, Paoletti, Stevens, & Hobson, 2016). Moore et al. (2016) suggested that students' meanings for graphs (such as graphs starting on the vertical axis, being read or drawn left-to-right, and passing the vertical line test) inhibit students from re-presenting images of the phenomenon that include covariational relationships. When students held these meanings for graphs they re-presented imagery that was distinct from how they initially imagined the quantities to covary in the situation. In this paper I extend Moore et al.'s (2016) work by exploring how the images a student constructs of the phenomenon influence both the graph the student constructs as well as the meanings the student constructs from that sketched graph. More specifically, I characterize two university precalculus students' graphing schemes to study the relationship between how the student initially understands the quantities to covary, the understandings he/she re-presents when constructing the graph, and the understandings he/she constructs from his/her completed graph.

Background

Covariational reasoning is "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354). Thompson and Carlson (2017) leveraged past research on variational and covariational reasoning to propose six major levels of covariational reasoning (see Figure 1) that are not constrained to reasoning about specific function types or methods of representation. Thompson and Carlson explain that the level of a students' covariational reasoning depends on three constructions: (1) the quantities the student is conceptualizing, (2) how the student imagines those quantities to vary, and (3) how the student coordinates and unites two changing quantities both in thought and representation. I elaborate on these three constructions in the following paragraph.

Major Levels of Covariational Reasoning	
Level	Description
Smooth Continuous Covariation	The person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and they envision both variables varying smoothly and continuously.
Chunky Continuous Covariation	The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation.
Coordination of Values	The person coordinates the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y).
Gross Coordination of Values	The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases". The person does not envision that individual values of quantities go together. Instead the person envisions a loose, non-multiplicative link between the overall changes in two quantities' values.
Pre-coordination of Values	The person envisions two variables' values varying, but asynchronously, one variable changes, then the second variable changes, then the first, etc. The person does not anticipate creating pairs of values as multiplicative objects.
No Coordination	The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values.

Figure 1: Thompson and Carlson's Major Levels of Covariational Reasoning, highest to lowest (Thompson and Carlson, 2017, p. 23)

Thompson (1990, 2011) explained that a quantity is a mental construction of a quality of an object that one can imagine measuring. Students construct quantities by conceptualizing an attribute to be measured and the way in which they would measure it. How the student imagines each quantity to vary constitutes her variational reasoning[1]. A student's conception of time is closely related to her variational reasoning since imagining a quantity's measure to change necessarily involves imagining time elapsing. Thompson (2011) described two ways students conceptualize time: experiential time "the experience of time passing" and conceptual time "an image of measured duration" (p. 27). Both experiential time and conceptual time are essential to covariational reasoning. For example, to construct what Castillo-Garsow (2012) calls smooth images of change one must imagine change in progress so that she imagines a quantity changing in her experiential time. Conceptual time, on the other hand, is essential to coordinate two quantities' measures at distinct moments in time (Thompson, 2011). The final construction Thompson and Carlson (2017) describe is the construction of a multiplicative object. As Saldanha and Thompson (1998) explained, a multiplicative object is a cognitive construction that enables one to hold two quantities in mind *simultaneously*. If one has coordinated two varying quantities through a multiplicative object then she anticipates that as one quantity changes the other quantity is changing as well. As a result, the student is able to hold both quantities in mind as they change together.

Theoretical Perspective

[1] See Thompson and Carlson (2017) for description of six major levels of variational reasoning.

According to Piaget (1967, 1985), actions are the source of all knowledge. Individuals organize their actions into schemes that include when to apply the action, an anticipation of the result of acting, how these actions work together, and eventually how these actions can chain together. As one engages in mathematical thinking he activates different scheme(s) in order to make sense of the task.

In mathematics, students are often asked to re-present their mathematical activity in the form of diagrams, graphs, formulas, tables, etc. If the student understands the graph (or formula, table, etc.) to be a depiction of his thinking then the student has an image of the mathematical activity he re-presented and the graph is a *representation* of that image. I emphasize the distinction between re-presenting and representing to be able to study student's graphing activity in the case the student does not anticipate he is representing, or creating a picture of, his mathematical thinking to then reason about.

Methodology

The subjects in this study were three university students: Ali, Bryan, and Sue. At the time of the study these students had recently completed precalculus but had not yet taken calculus. These students were selected to participate in the study because they collectively demonstrated different ways of engaging in covariational reasoning in a recruitment interview (see Frank, 2017 for more details on recruitment and selection). After being selected, each student participated in a two-hour one-on-one task based clinical interview (Clement, 2000). The purpose of the clinical interview was to characterize each student's meanings for graphs.

After completing the interview process I engaged in retrospective analysis by identifying instances that provided insights into the relationship between the understandings the student re-presented when constructing a graph and how the student understood his/her sketched graph. I used these instances to generate tentative models of each student's schemes for graphing. I tested these models by searching for instances that confirmed or contradicted my model and repeatedly refined my model until it accounted for the student's mathematical activity.

Results

Of the three students who participated in the study two students (Ali and Bryan) conceptualized graphs in terms of varying quantities. Sue, on the other hand, conceptualized graphs as pictures of an object's motion (consistent with Monk's (1992) notion of *iconic* translations). In this section I describe how Ali and Bryan imagined quantities to covary when reasoning about a situation, when constructing a graph, and when reasoning about their sketched graphs.

Pre-Coordination of Values: The Story of Ali

When Ali created a graph from a contextual description of a situation she engaged in two distinct activities. First, Ali generated a shape by tracking one quantity's variation as she imagined that variation in her experiential time. Then, Ali used the properties of the shape she created to reason asynchronously about the variation of the two quantities labeled on the graph's axes. If the shape she created did not match her anticipation of how each quantity varied, then she guessed shapes from her memory of past graphing activities until she picked a shape that matched how she imagined each quantity to vary. This suggests that Ali used distinct and uncoordinated systems of actions when generating graphs (drawing shapes) and understanding her sketched graphs (reasoning about two quantities' asynchronous variation). I will illustrate Ali's graphing scheme with her engagement in the skateboard task (see Figure 2).

A skateboarder skates on a half-pipe like the one shown. The skateboarder goes across the half-pipe and then returns to the starting position.

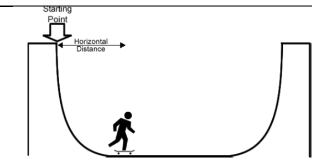
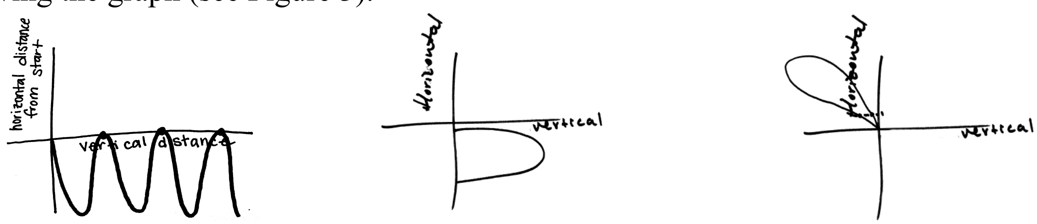


Figure 2: Description of skateboard task.

I asked Ali to graph the skateboarder's horizontal distance to the right of the starting position relative to the skateboarder's vertical distance above the ground. Ali made three attempts drawing the graph (see Figure 3).



Ali's first attempt
Ali's second attempt
Ali's third attempt

Figure 3: Ali's three attempts to graph skateboarder's horizontal distance from start relative to his vertical distance above the ground.

On Ali's first attempt she drew an oscillating curve in the fourth quadrant (Figure 3). Since Ali imagined the half-pipe below ground, it seems Ali made this graph by tracking how she imagined the skateboarder's vertical distance changing as she imagined the that variation in her experiential time. After drawing the curve, and without prompting, Ali determined her graph was incorrect because "the graph I drew is showing that the vertical distance is increasing the whole time." She went on to draw two more shapes (Figure 3) and each time appropriately reasoned why her sketched graph was incorrect. For example, Ali rejected her second attempt (a side-ways U-shape in the fourth quadrant) since it showed the vertical distance was positive when she wanted to show the vertical distance was negative. After Ali rejected her third graph I asked her to explain her approach to graphing (Excerpt 1).

Excerpt 1: Ali's explanation of making graphs by guessing and checking shapes

Int: What are you doing when you are trying to figure out what graph it could be?

Ali: Um. Well I think of like. I either focus. I go back and forth with like okay vertical distance and horizontal distance. So I think of potential like, I guess shapes, that can be drawn and then I'm like does this fit the characteristic of the horizontal distance. If it doesn't then it is out and I think of another one. And so. That's how I usually go about with graphing graphs until I eventually - I'm like this one fits both criteria.

In Excerpt 1, Ali describes her three-step approach to graphing: (1) draw a shape by "think[ing] of potential … shapes that can be drawn", (2) consider what the shape conveys about the variation of each quantity *separately*, and (3) adjust the shape until it matches how she imagined each quantity to vary. This final step is significant because it implies Ali constructed two distinct images of the quantities' covariation; Ali constructed an image of each quantity's variation from the graph that she compared to her image of each quantity's variation from her understanding of the situation. This suggests Ali had an image of the quantities' variation that she could have re-presented when making her graph.

I hypothesize Ali did not make her graph by re-presenting the images of varying she constructed from the phenomenon because she attended to each quantity's variation separately, what Thompson and Carlson (2017) called a *pre-coordination of values*. For example, Ali attended only to the skateboarder's vertical distance when determining the validity of her first and second graphs and attended only to the skateboarder's horizontal distance when determining the validity of her third graph. Additionally, in Excerpt 1 Ali explained that she "go[es] back and forth with like vertical distance and horizontal distance…like does this one fit the characteristic of horizontal distance". I take this as evidence that Ali understood the shape of a graph to show how each quantity varied separately.

By imagining each quantity's variation separately I claim that Ali did not have a single image from having coordinated two quantities' variation that she could attend to when making her graph. In other words, Ali did not have a way to think about making one shape that would convey how the skateboarder's horizontal distance changed *and* how the skateboarder's vertical distance changed. Instead, Ali was constrained to making a graph by re-presenting only one of her images of variation (first attempt in Figure 3) or guessing and checking shapes (second and third attempt in Figure 3). In summary, it seems Ali's asynchronous coordination of the two quantities' variation inhibited her from attending to both quantities' variation when making her graph.

Coordination of Values: The Story of Bryan

Like Ali, Bryan demonstrated different conceptualizations of the varying quantities when constructing his graph and when reasoning about his graph. More specifically, Bryan constructed graphs by re-presenting his experience imagining a continuously varying quantity but he did not reason about his graph in terms of a quantity's continuous variation. Instead, he reasoned about his graph by coordinating static states in each quantity's variation. I will illustrate Bryan's graphing scheme with his engagement in the bottle evaporating task[2].

In the bottle evaporating task I asked Bryan to imagine a spherical bottle filled with water that was left outside to evaporate. Then I asked him to graph the height of water in the bottle relative to the volume of water in the bottle as the water evaporated. Before Bryan constructed a graph he reasoned, "When volume is maximum the height should be maximum and when volume is zero height should be zero." This suggests Bryan coordinated two quantities' measures at two moments in time. He proceeded to draw a straight line from the top middle of the plane that fell from left to right (see Figure 4, red line).

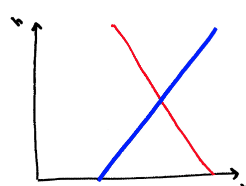


Figure 4: Bryan's initial (red) and revised (blue) graph for the evaporating water problem (task adapted from Paoletti & Moore, 2016)

[2] Bottle evaporating task from Paoletti and Moore (2016).

From my perspective, the line Bryan drew was not a re-presentation of the pairs of measures he imagined in the situation. Instead, it seems Bryan made his initial point with the anticipation of showing the simultaneous state of maximum height and maximum value. Then he drew a line by imagining the height of the water decreasing as he imagined the water in the bottle evaporating. This suggests that Bryan constructed his graph by imagining the gross variation of the height of the water in the moment he imagined that variation in his experiential time.

After Bryan drew the line he reconstructed his initial image of two pairs of quantities' measures to reason that his graph should show maximum height and maximum volume. He determined that his graph did not show this relationships saying, "It doesn't make sense. Because over here (*points to start of line in top middle of plane*) it says height is maximum but volume is not maximum (*points to intersection of line with horizontal axis*)." Bryan drew a new graph that was a vertical reflection of his original graph about its midpoint; his graph now decreased from right to left (see Figure 4 blue line). Bryan explained that now he understood his graph to show the height is maximum when the volume is maximum and also show the height is minimum when the volume is minimum.

In summary, Bryan engaged in three distinct activities when completing the bottle evaporation task. First he imagined each quantity's (discrete) variation and coordinated the two varying quantities by constructing pairs of measures, a point's coordinates. Then he drew a line by re-presenting his experience attending to one quantity's gross variation as he imagined it changing in his experiential time. Finally, he reconstructed his initial image of pairs of quantities' sizes to determine if the behavior of the sketched graph matched his anticipation of the relationship between the quantities' measures.

I hypothesize that Bryan did not make his graph by re-presenting his initial image of pairs of measures because he could not anticipate creating these pairs of measures as he imagined a quantity to continuously vary in his experiential time. In other words, it seems that Bryan needed to imagine a static state in the quantities' variation in order to coordinate two quantities' measures. As soon as he imagined one quantity's measure to change he could no longer coordinate two quantities' measures. This implies that the way Bryan coordinated two varying quantities inhibited him from re-presenting his understanding of how two quantities change together when making his graph.

Discussion

Ali and Bryan both demonstrated different images of covarying quantities when making a graph and when reasoning about that sketched graph. While this highlights the meanings students learn to impose on the products of their graphing actions, the findings from this study suggest that the meanings students construct from their sketched graph are consistent with how they imagined the quantities to covary in the situation. In the examples above, Ali reasoned separately about two quantities' smooth variation both when reasoning about the situation and when reasoning about her graph. Similarly, Bryan reasoned about pairs of measures both when reasoning about the situation and when reasoning about his graph. This suggests that while a student might have distinct experiences making a graph and reasoning about that graph these experiences are actually governed by the same scheme. More specifically, the student's activity making a graph is the result of an accommodation to their scheme for covariational reasoning in order to have actions available to them that persist under variation. For both Ali and Bryan this accommodation involved attending to one quantity as she/he imagined it changing in her/his experiential time. This is significant because it implies that students engage in different levels of covariational reasoning throughout their graphing activity because they are unable to re-present how they initially imagined the quantities to change together.

This study provides evidence that the nature of the student's coordination can inhibit him/her from re-presenting his/her understanding of how the quantities covary in the situation. For example, Ali coordinated two quantities' variation by imagining each quantities' variation separately. As a result, she did not have a single coordinated image to attend to when making her graph. Ali anticipated that she could use whatever shape she made to see the variation of each quantity, but she did not have a way to think about how to make that shape. Instead, she made her graph by guessing shapes until she picked one that appropriately matched how she imagined each quantity to vary.

Bryan coordinated two quantities' variation by coordinating static states in each quantity's variation and constructing the coordinates of a point in the Cartesian plane. However, as soon as he imagined one of the quantities to vary he no longer had an image of a static state in which he could coordinate two measures. As a result, when he attempted to construct his graph he did not continuously coordinate quantities' measures. Instead, Bryan made his graph by imagining one quantity changing in his experiential time. After making his graph, however, Bryan imagined coordinating measures to reason about what his sketched graph represented; he appeared to reason about an infinite collection of points on his graph. In summary, since Bryan's image of plotting points did not persist under variation, Bryan could not re-present his coordination of the quantities' variation as he imagined a continuously changing phenomenon.

Researchers repeatedly emphasize the importance of holding two quantities in mind when constructing a graph (e.g., Moore et al., 2016; Whitmire, 2014). This study provides further evidence that this is a nontrivial construction. More specifically, students need to construct ways to organize their images of varying quantities so that they can hold two quantities in mind as they imagine both quantities to change. I hypothesize that if students hold both quantities in mind then they have something new to represent in a graph – namely the coordination of two quantities. Teaching experiments with Ali and Bryan (see Frank, 2017) suggest that conceptualizing a point as a correspondence point, imagining a graph being made of Tinker Bell's pixie dust, and imagining the phenomenon happening in little chunks (e.g., taking baby steps) might support students in coordinating their images of varying quantities[3] and re-presenting this coordination in a graph .

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[3] Correspondence point didactic object from Thompson, Hatfield, Yoon, Joshua, and Byerley (in press) and Tinker Bell didactic object from Thompson (2013)

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