Future Middle Grades Teachers' Coordination of Knowledge Within the Multiplicative Conceptual Field

Andrew Izsák and Sybilla Beckmann
The University of Georgia

We report theoretical and empirical results generated through studying several cycles of a number and operations content course offered to future middle grades mathematics teachers. A main feature of the course is using an explicit, quantitative definition for multiplication to connect a range of topics in the multiplicative conceptual field (Vergnaud, 1983, 1988). Course topics include multiplication and division with both whole numbers and fractions, proportional relationships, and linear functions. The theoretical results include a mathematical analysis of multiplication as coordinated measurement and a (still emerging) psychological framework that emphasizes coordinating diverse cognitive resources. Empirical data come from clinical interviews conducted with 6 future teachers enrolled in the content course in Fall 2016. One empirical result is the importance of connecting partitioning quantities, dividing measurements by whole numbers, and multiplying measurements by unit fractions when expressing relationships between quantities through multiplication expressions and equations.

Keywords: Quantitative Reasoning, Multiplication, Equations

Improving instruction in topics related to multiplication remains a central challenge for mathematics education. For purposes of the present report, we consider multiplication and division with whole numbers and fractions, proportional relationships, and linear functions of the form $y = \mathrm{m}x$. The importance of these topics has been emphasized by curriculum standards (e.g., National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics, 2000) and national reports (e.g., Center for Research in Mathematics & Science Education, 2010; National Mathematics Advisory Panel, 2008). Nevertheless, despite several decades of research, the topics listed above pose perennial challenges for both students and teachers, and difficulties with these topics can be a primary obstacle to college readiness (e.g., National Center on Education and the Economy, 2013). The present report comes from an on-going NSF-funded study in which we are investigating ways to help future mathematics teachers develop integrated and coherent understandings of topics related to multiplication.

Background

We draw on Vergnaud's (1983, 1988) construct of the *multiplicative conceptual field* (MCF) that consists of "all situations that can be analyzed as simple and multiple proportion problems and for which one usually needs to multiply or divide" (Vergnaud, 1988, p. 141). Vergnaud included whole-number multiplication and division, fractions, ratios and proportions, linear functions, and further topics in the MCF.

Most research on teachers' understandings of the MCF has concentrated on deficits with respect to particular topics. Although many teachers can use algorithms to determine the product of two fractions or decimals, a host of studies (e.g., Behr, Khoury, Harel, Post, & Lesh, 1997; Eisenhart et al., 1993; Graeber & Tirosh, 1988; Graeber, Tirosh, & Glover, 1989; Harel & Behr, 1995; Sowder, Philipp, Armstrong, & Schappelle, 1998; Tirosh & Graeber, 1990) have reported constraints on in-service and preservice teachers' performance when explaining products of fractions or decimals embedded in problem situations. Similarly, although many U.S. teachers can compute the quotient of two fractions or decimals using algorithms, they often experience difficulties explaining division when it is embedded in problem situations (e.g., Ball, 1990; Borko et al., 1992; Graeber & Tirosh, 1988; Jansen & Hohensee, 2016; Lo & Lou, 2012; Ma, 1999; Simon 1993; Tirosh, 2000).

The small handful of studies on teachers' capacities to reason about proportional relationships report that middle grades teachers perform poorly on test items that, ideally, their students should be able to solve (Post, Harel, Behr, & Lesh, 1991). Teachers can have difficulty distinguishing missing-value problems that ask about proportional relationships from ones that do not (e.g., Cramer, Post, & Currier, 1993; Fisher, 1988; Lim, 2009), can have trouble coordinating two quantities in a proportional relationship (e.g., Orrill & Brown, 2012), can make additive comparisons inappropriately (e.g., Canada, Gilbert, & Adolphson, 2010; Lim, 2009; Son, 2010), and can have trouble conceiving of a ratio as a measure of a physical attribute, such as steepness or speed (Simon & Blume, 1994; Thompson & Thompson, 1994). With respect to problem-solving strategies, teachers can rely heavily on cross multiplication or other formal methods (e.g., Fisher, 1988; Harel & Behr, 1995; Orrill & Brown, 2012), guess at arithmetic operations (Harel & Behr, 1995), and search for key words (Harel & Behr, 1995).

Theoretical Frame

In contrast to the numerous studies that have emphasized deficits in teachers' understandings of particular topics included in the MCF, in our project we concentrate on emerging competence characterized as developing a coherent perspective that connects various topics related to multiplication. Our conjecture is that teachers might better understand individual topics within the MCF by developing a single lens that ties them together. The framework we present for such an integrated understanding combines mathematical and psychological perspectives.

Figure 1 shows the quantitative definition of multiplication upon which we have converged. It applies to situations in which there is a quantity (the product amount) that is simultaneously measured with two different measurement units (a "base unit" and a "group"). The most important aspects of this definition are (a) writing the multiplicand and multiplier in a consistent order to support a coherent view of multiplication, division, and proportional relationships (e.g., Beckmann & Izsák, 2015) and (b) interpreting N, M, and P in Figure 1 as numbers that result from measuring quantities in terms of some designated unit.[1] N and P refer to measuring with base units, and M refers to measuring with groups.

N	•	M	=	P
How many base units make one group exactly?		How many groups make the product amount exactly?		How many base units make the product amount exactly?

Figure 1. A quantitative definition for multiplication based in measurement.

The definition in Figure 1 can be used to coordinate an important swathe of the MCF—for instance, by viewing division as multiplication with an unknown factor and proportional relationships as instances where values for two of N, M, and P co-vary while the value for the

[1] Our emphasis on numbers arising from measuring quantities in terms of designated units is consistent with aspects of Thompson's (2010) discussion of quantitative reasoning.

third remains fixed (Beckmann & Izsák, 2015). The definition in Figure 1 is also consistent with the definition for fractions found in the Common Core State Standards for Mathematics (CCSS; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) that presented the fraction A/B and A copies of the unit fraction $1/B$. Figures like that shown in Figure 2a can support the measurement perspective on unit fractions if one asks how many of the long strip make the short strip exactly (1/3). We have found that future teachers have little problem answering such questions and can extend this measurement perspective from unit fractions to non-unit fractions (Figure 2b). This appears to be a reliable foothold for future teachers when extending the measurement definition of multiplication shown in Figure 1 from whole numbers to fractions.

Figure 2. (a) Interpreting 1/3 from a measurement perspective: 1/3 of the long strip makes the short strip exactly. (b) Interpreting 2/3 from a measurement perspective: 2 (1/3 of the long strip) make the short strip.

Our psychological perspective is informed by diSessa's (1993, 2006) knowledge-in-pieces epistemology. Knowledge-in-pieces is a constructivist perspective in which learners come to know by using and refining knowledge as they construct interpretations of their interactions with the physical and social environment. The perspective characterizes the evolution from novice to expert knowledge as piecemeal construction, refinement, and reorganization of diverse fine-grained knowledge resources that are connected to varying degrees and whose use is often sensitive to context. Examples of cognitive mechanisms include refining the contexts in which resources are applied, forming new connections among resources, and loosening connections among others. In the present study, we examined the ecology of resources that future middle grades teachers used as they coordinated the definition of multiplication shown in Figure 1 with diverse problem situations that are contained in Vergnaud's (1983, 1988) MCF. Past research has used the knowledge-in-pieces perspective to demonstrate that coming to see diverse problem situations through a common lens can be a significant accomplishment (e.g., Wagner, 2006).

Methods

In Fall 2016, we recruited six future middle grades teachers who were enrolled in a 2-semester sequence of mathematics content courses. The second author taught both courses. Both courses made extensive use of the definition of multiplication shown in Figure 1. The first course (Number and Operations) covered multiplication and division with whole numbers, the CCSS definition for fractions, the meaning of the equal sign, reasoning from definitions, multiplication with fractions, partitive and measurement division with whole numbers, and connecting division to fractions. The second course (Algebra) focused on proportional relationships, linear equations, and further topics. Teachers in the course were invited to participate in interviews, and the six were selected based on performance on a fractions survey administered the first week of the Number and Operations course.

This report focuses on the first three interviews we conducted during the Number and Operations course. The interviews were spaced a few weeks apart and were coordinated with whole-class instruction, most often so that the interviews provided information about the future teachers' reasoning before specific topics were introduced in the course. The first interview examined how future teachers thought about multiplication as a model of problem situations and how they formed equations of the form $y = \mathrm{m}x$ before the definition of multiplication shown in Figure 1 was introduced in the course. The CCSS definition of fractions had already been introduced and the interview tasks included fractional multipliers and multiplicands. The second interview took place after the definition in Figure 1 was introduced (only with whole numbers) and was designed to access future teachers' facility with the mental operations of splitting and units coordination that have emerged as important in research on children's fractional knowledge (e.g., Steffe, 2003) and also how they formed equations of the form $y = \mathrm{m}x$ at this point in the course. The third interview took place after instruction in fraction multiplication, division as multiplication with unknown factor, the distinction between partitive and measurement division in the context of whole numbers, and the connection between division and fractions. The third interview was designed to see how future teachers reasoned about division in the context of proportional relationships and linear equations, topics that would be covered in the subsequent algebra course. Many of the interview tasks asked future teachers to solve problems using a math drawing. Examples of such drawing include number lines and tape or strip diagrams.

We recorded all of the interviews with two cameras—one focused on the interviewer and research participant and one focused on written work—and collected all of the written work generated during the interviews. A third party transcribed the interviews verbatim. The present report is based on analysis of talk, gesture, and inscription as captured in the videos, transcripts, and written worked generated during the interviews. (In addition to the interview data, we also collected the participants' homeworks, quizzes, and tests assigned in the course.)

We analyzed talk, gesture, and inscription line-by-line for evidence of the knowledge resources that the future teachers appeared to employ. We wrote analytic notes to capture our interpretations of how future teachers were reasoning moment-to-moment. The notes included observations about similarities and differences both within a given teacher across different tasks and across different teachers on the same task. In some cases, we took future teachers' statements as direct and reliable reports of their thinking. In other cases, we made inferences about aspects of future teachers' reasoning that they would not likely be able to report directly.

Results

Future teachers in the present study employed a complex ecology of cognitive resources when working on tasks across the interviews. To illustrate our results, we make three comments about that ecology that span all six participants and then provide more detailed description of one participant.

First, during the interviews, the future teachers demonstrated facility with whole-number factor-product combinations, algorithms for multiplying fractions, and cross multiplication for solving proportions. We assumed that when future teachers employed these resources, they drew on what they remembered from their K-12 mathematics education. Such resources can be viewed in a negative light when they interfere with reasoning about quantities directly. Although we did observe cases where future teachers determined numerical answers through computations, and thereby circumvented reasoning with quantities, we also observed cases in which future teachers used calculation constructively when solving and explaining problems in terms of math drawings. These data suggest that resources for numerical calculation are not necessarily in opposition to resources for reasoning with quantities but rather could be part of a larger ecology in which numerical calculation and reasoning with quantities support one another.

Second, the future teachers expressed a variety of meanings for multiplication and the equal sign. Meanings for multiplication included widely known ones such as multiplication is about repeated groups and that, in the case of fractions, "of means multiply." For these future teachers, repeated groups and "of" oftentimes appeared to be two disconnected understandings of multiplication rather than different expressions of a single, unified conception of the operation (such as the one shown in Figure 1). Meanings for the equal sign also included several well-known ones, such as an indication to complete a computation, an indication that two numbers co-occur (leading to equations that actually express ratios and look like the classic student-professor error), and the number on the left-hand side is the same as the number on the right-hand side. In one interview, we saw one participant encounter difficulties when he used all three of these meanings for the equal sign when working on a single problem. More generally, multiple meanings for multiplication and the equal sign often led to piecemeal reasoning across tasks which used different combinations of whole numbers and fractions for the multiplier (M in Figure 1) and multiplicand (N in Figure 1). These data suggested that sometimes future teachers experienced challenges during the interviews not so much because they lacked a particular cognitive resource but rather because they had trouble recognizing when some of those resources might be more useful than others. Knowledge-in-pieces' emphasis on knowledge refinement is well-suited to handle such phenomena.

Third, future teachers employed to varying degrees two mental operations on quantities highlighted in past research on children's fractional knowledge. These are splitting (e.g., Steffe, 2003) and different levels of units coordination (e.g., Hackenberg, 2010). Steffe's splitting operation is a fusion of partitioning and iterating. We asked versions of splitting tasks that made explicit connections to the measurement sense of unit fractions illustrated in Figure 2. None of the participants had difficulty with our splitting tasks. In particular, when presented with a strip like that shown in Figure 2a and told that the strip was a whole number of times (in actual interviews we numbers like 8) longer than a second strip, future teachers had no trouble constructing the second strip or explaining how many of the long strip made the short strip exactly. To illustrate, in case of a diagram like that shown in Figure 2a, all future teachers could explain that 1/3 of the long strip made the short strip exactly. Although, with an appropriate prompt, all the future teachers could express a measurement perspective on unit fractions, we saw differences in performance on tasks designed to elicit units coordination.

For the rest of our results section, we present examples of reasoning from Hanna around whole-number and fractional multipliers (M in Figure 1). We will sketch evidence that during the first interview Hanna was more proficient reasoning with and explaining whole-number multipliers than fractional multipliers, even though she also demonstrated a measurement perspective on fractions, and that during the second interview she had begun to coordinate a measurement sense of fractions with multipliers. The examples illustrate how mechanisms like coordination and refinement of fine-grained knowledge resources that are emphasized in the knowledge-in-pieces perspective are a good fit for reasoning we observed. In the full paper, we will present data from others of the six future teachers that also illustrate knowledge coordination and refinement and that provide perspectives on the multiplier and multiplicand that contrast with Hanna's.

We began the first interview with a set of word problems that described (to us) multiplication situations with different combinations of whole-number and fractional multipliers. This interview took place before meaning of multiplication shown in Figure 1 was introduced in the content course. During her first interview, Hanna solved a variety of problems with whole-number multipliers without difficulty, explaining that collecting objects into whole groups cued multiplication for her. For instance, she had no difficulty writing an equation that fit the tennis ball situation: *Jacinda has 4 cans of tennis balls. If there are 3 balls in a can, how many tennis balls does she have in all?* Hanna wrote 4 x 3 = 12 and justified her work as follows: "When you're dealing with different kinds of objects, it's easier to see with multiplication. Like if you had…instead of bags, if you had like 5 soccer balls and then another 5 soccer balls and they didn't even talk about bags, then I'd probably do 5 plus 5." A few moments later during the same interview, we asked her to write an expression or equation that fit the Chili situation: *Nick uses 1/5 of can* [sic] *of tomato paste in his chili recipe. The can contains 4 ounces of tomato paste. How many ounces of tomato paste does he use in his chili?* This time Hanna wrote "1(4) = 1/5 (X)" and explained that she was attempting to use ratios. She then stated that the problem was confusing because "1/5 is talking about of the can, 4 is talking about ounces. So those are two different things." Notice that Hanna attended to a similar feature of both situations—two different things—with very different results. These data provided initial evidence that she did not have a single conception of multiplication that she could apply across problems with whole-number and fractional multipliers.

In subsequent work, also from the first interview, Hanna demonstrated that she did not fully coordinate partitioning a quantity into equal-sized pieces, dividing the value of that quantity by a whole number, and multiplying the value of that quantity by a unit fraction. As she continued to work on the Chili task, she proposed dividing 4 ÷ 1/5 and explained:

> Because you're trying to find a part of the whole and like how much a part of the whole equals. So if you know…you know that the can is 4 ounces and you need…you want to find 1/5…how many ounces 1/5 of that can is then you need to divide 4 ounces by 1/5 to find out how many ounces there are in 1/5.

Later on during the first interview, Hanna worked a more complicated task: *One serving of oatmeal is 1/3 of a cup. For one meal, Chelsea are 2/5 of a serving. How many cups of oatmeal did Chelsea eat?* From the data we could not tell exactly why, but Hanna refined her connections between partitioning and numerical calculation. In particular, she offered a series of explanations, each of which included refinements to previous explanations. Her final explanation coordinated partitioning one serving into 5 parts with dividing 1/3 by 5. She wrote 2 (1/3 ÷ 5) and explained: "So first I divided my 1/3 cup into 5 pieces, because I know that she ate 2/5. So, if I divide it into 5 pieces and then multiply by 2, I can get my 2/5." A few moments later, Hanna acknowledged that she did not know how to compute 1/3 ÷ 5, which we took as evidence that she was not using results of this computation when refining her connections among partitioning, dividing, and actions that could support understanding fractions as multipliers.

Further evidence from the first interview that Hanna did not connect partitioning with multiplying by a unit fraction came on the Pebble task shown in Figure 3. Hanna had no trouble drawing the short path, defining P to be the "amount of pebbles in the long path," and expressing the number of pebbles in the short path with the expression $P \div 8$. At the same time, she explicitly rejected multiplication as a viable option:

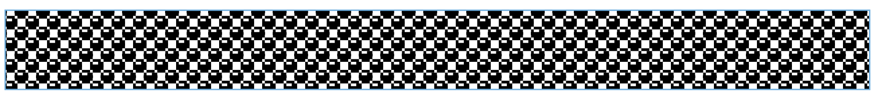
> I don't know how you would use multiplication, because you're trying to find out how much pebbles there are in 1/8 of this long path. So in order to get that, you have to divide the amount of pebbles by 8 to get, right? Yeah, to get that one amount of pebbles. To get…I don't know how you would you use multiplication honestly.

The second interview took place about 3 weeks after the first and after the definition of multiplication shown in Figure 1 had been introduced in the content course. At the time, the definition had only been used with whole numbers. During the second interview, Hanna demonstrated facility with measurement that went beyond her work during the first interview. When working on the task shown in Figure 4, she defined b to be the "beads in short strip" and B to be "beads in long strip." She then generated and explained the equation $B = b$ (LS/SS):

> So first I'm going to see how many short strips can go into my long strip. So I divide my long strip by my short strip, and then I'm going to multiply whatever I get. So how many short strips can go into my long strip times the amount of beads that are in my short strip, and then…then you get how many beads are in your big strip.

Notice first that Hanna appeared to apply a measurement sense to the fraction notation, LS/SS, when discussing "how many short strips can go into my long strip" and second that she connected this measurement sense to the multiplier. Further evidence from the second interview made clear that, although Hanna had made important steps in the right direction, she still had further to go when thinking about fractional multipliers.

The drawing below shows a 1 kilometer long garden path. It is 8 times as long as another garden path. Please draw the other path.



Imagine that the same two paths are covered with pebbles that are of uniform size and are spread evenly on both paths. The dots shown in the long strip indicate the pebbles.

If you knew the amount of pebbles covering the long path, how could you express the amount of pebbles covering the short path?

Figure 3. The Pebbles task from Interview 1.

Some strips of fabric have tiny beads sewn onto them. The beads are spread uniformly across each strip.

If you knew the amount of beads in the short strip, how would you find the amount of beads in the long strip?

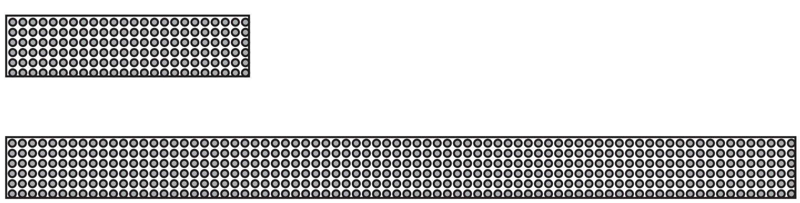


Figure 4. The Beads task from Interview 2.

Conclusion

We are still a long way from illustrating how teachers might better understand individual topics within the MCF by developing a single lens that ties them together; but, the example of Hanna suggests processes through which teachers might construct such a lens. In particular, using the example of emerging facility with fractional multipliers we have illustrated how Hanna used fine-grained coordination and refinement of knowledge resources to extend her understanding of multipliers in the case of whole numbers.

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