Revisiting Reducing Abstraction in Abstract Algebra

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In this paper, we revisit Hazzan's (1999) fundamental work on reducing abstraction in abstract algebra tasks. As we analyzed hundreds of students' activity related to abstract algebra tasks, we identified many ways students reduced abstraction that did not align with the original framework. We leverage additional theories of abstraction to expand and refine Hazzan's framework to reflect new aspects of familiarity, contextualization, complexity and connectedness, and formality. For each of the new categorizations, we provide illustrations of students engaged in the relevant reduction of abstraction. We conclude with consideration to how the expanded framework may highlight productive types of abstraction reduction.

Keywords: Abstraction, Abstract Algebra, Student Activity

It is well-documented that abstract algebra is a challenging course for students (Dubinsky, Dautermann, Leron, & Zazkis, 1994; Leron, Hazzan, & Zazkis, 1995; Weber & Larsen, 2008). For many students, this is the first time they engage with mathematical objects that are brought into existence via formal definitions. These stipulated concepts are general, complex, and often unfamiliar to students. Hazzan (1999) created the *reducing abstraction* framework to document how students engaged with the generality, complexity, and unfamiliarity of concepts in abstract algebra tasks. She leveraged a number of theories of abstraction to categorize various ways students reduced abstraction when engaging these tasks. This work is foundational and remains one of the more nuanced treatments of student activity in abstract algebra.

In our recent work exploring hundreds of students responses to abstract algebra tasks (Melhuish, 2015), we similarly observed students reducing abstraction. However, we identified a number of ways students reduced abstraction beyond the classifications in Hazzan's (1999) work. In this paper, we synthesize additional theories of abstraction to expand Hazzan's framework in order to better reflect the nuances and variety of approaches found in our students' activity. We share our expansions and provide illustrations of students engaged in reducing abstraction in both productive and unprodctive ways.

Theories of Abstraction in Mathematics Education

In the field of mathematics education, we have many treatments of the abstraction construct stemming from Piaget's comprehensive work to von Glasersfeld's constructivism and Freudenthal's *Realistic Mathematics Education*. As Piaget noted (1980), "All new knowledge presupposes an abstraction..." (p. 89). However, what scholars mean by an abstract concept, and what we mean by abstraction varies according to a given theory of learning. Hazzan (1999) originally identified three treatments of abstraction: relationship between the object of thought and the thinking person, process-object duality, and complexity of concept of thought. We see these three categorizations as essential, but not exhaustive for exploring student task engagement in the setting of abstract algebra. We discuss several theories of abstraction that ultimately inform our expanded framework.

Before we begin the discussion, we acknowledge an important dimension along which theories of abstraction differ: activity-based versus cognitive. In our overview, we condense features of the theories with little attention to whether the theory was meant to describe cognition or activity. Rather, our purpose is to identify the means through which abstraction is posited to occur.

Abstracting via Apprehending Properties

A number of abstraction theories focus on students apprehending properties from a set of known objects. Piaget's (2013) theory of empirical abstraction provides the foundation of much of this work. For empirical abstraction, properties are observed through empirical investigation. If you view a set of white objects, you can abstract the idea of whiteness. Skemp (1986) further expanded this theory explaining, "Abstracting is an activity by which we become aware of similarities ... among our experiences. Classifying means collecting together our experiences on the basis of these similarities" (p. 21) Skemp presented a two-part process of recognizing similarity and then creating a class of object based on similarities. Scheiner (2016) built on this idea further by introducing structural abstraction. Rather than purely empirical (abstracting from empirical objects), abstraction can occur through exploration of mental objects. This exploration may be focused on similarity, but may also occur through focusing on complementary aspects. In each of these theories, a concept is abstracted through collecting a relevant set of properties.

Abstracting via Building Connections and Complexity

An alternative lens for abstraction focuses on building connections between or within concepts. Connections play a fundamental role in a number of abstraction theories such as within Dubinksy and McDonald's (2001) schemas or Hoyles, Noss, and Kent's (2004) webbing. Abstraction occurs through the correct coordination of various concepts. This may be internal such as in Dayvdov's (1990) theory where understanding a concept involves unity amongst its connected parts. Alternately, an assembly metaphor (e.g. Ohlsson and Lehtinen, 1997) may underlie a connection focused abstraction theory. Ohlsson and Lehtinen explained that new knowledge structures are developed via assembling "previously acquired ideas" (p. 42). In this sense, a concept is abstracted via coordination of various properties and/or concepts that compose the finalized object.

Abstracting via Decontextualization

Decontextualization theories tend to focus on moving from a familiar context to building something abstract that is independent of the context. This type of abstracting can be found in the school of Realistic Mathematics Education and Hershkowitz, Schwarz, and Dreyfus' (2001) *abstraction in context.* These theories distinguish horizontal mathematizing, "the process of describing a context problem in mathematical terms – to be able to solve it with mathematical means" (Gravemeijer & Doorman, 1999, p.117), from vertical mathematizing, where this activity is mathematized through abstracting, generalizing and formalizing (Rasmussen et al., 2005). This type of abstraction occurs when *model of* a specific context or problem (one which is mathematically real to a student) transition to a *model for* additional mathematics that does not rely on the underlying context. These task-based theories align themselves with two views of abstraction related to familiarity. First, a concept can be thought of as abstract if it has moved from a model of a familiar situation to a model for other contexts. Alternately, a concept within a context is more or less abstract depending on how mathematical real it is to an individual. This is roughly equivalent to a student's familiarity with it (cf. Wilensky, 1991).

Abstracting via Delineation and Refinement

While many theories posit that concepts move from concrete to abstract, Dayvdov (1990) introduced an alternative view where the abstraction process concretizes an abstract kernel of an idea. His theory posits that an object begins as an undeveloped (potentially inconsistent) basic form. This form can be analyzed, and refined until a coherent model is developed. This theory of abstraction can be thought of as moving from a vague idea of concept to a concretely delineated defined concept. The delineation may be more fundamental in advanced mathematics where stipulated definitions form the basis of mathematical structures. Zandieh and Rasmussen (2010) provide insight into this sort of refinement through illustration of students' concept images and definitions of triangle developing. In some senses, this type of abstraction connects to pseudo-empirical abstraction (Piaget, 2013) where abstraction can occur via interacting with an object. In Dayvdov's sense, an object may be a mathematical model rather than a purely empirical "real-world" object. Tall and Pinto (2002) provide such an example where a student moves from a generic visual representation of limit to build to the formal definition. From this theoretical lens, a concept is abstracted when a stipulated definition is abstracted from imprecise models.

Abstracting via Encapsulating Processes

The final treatment of abstraction is that of process-object duality. This type of abstraction has been explicated through a number of theories including Dubinsky and McDonald's (2001) Action-Process-Object-Schema theory, Sfard's (1991) object reification, and Gray and Tall's (1994) procept theory. Each of these theories operationalizes Piaget's work in the context of various mathematics settings. The underlying feature is the encapsulation of or reification of some particular process into an object. These theories break into three stages: a process that requires individual steps, a holistic view of the process, and a view of the process as an object itself to be used in other processes. For example, Asiala et al. (1997) illustrated this duality in abstract algebra where students may rely on the canonical procedure for creating a coset rather than or in conjunction with treating a coset as an object itself. In this sense, a concept is abstracted when it is no longer treated exclusively as a process, but rather can be used as an object for other processes.

In synthesizing the preceding theories, abstraction has a dual nature: it can be seen both as a cognitive activity and as the concept resulting from that activity. When viewed as a cognitive activity, *abstraction* is a process that transforms a concept via given means. The resulting concept is said to be an abstraction (or "abstracted"). In what follows, we use the term "level of abstraction" to refer to the means by which the student carries out the abstracting activity. Thus, in *reducing abstraction*, an individual is acting cognitively via specific means in order to reduce for herself the level of perceived abstraction. Reduction is tied to the specific context in which the student is working. The Expanded Reducing Abstraction framework in Table 1 presents the levels of abstraction and operationalizes the means by which the activity is carried out.

Reducing Abstraction: An Expanded Framework

We leverage the prior discussion of abstraction theories to introduce an expanded classification of reducing abstraction. As in Hazzan's (1999) work, we do not claim that these ways of reducing abstraction are mutually exclusive or exhaustive. Rather, we introduce the framework as a tool for making sense of the many ways students engage with tasks containing abstract concepts. We illustrate categorization with data from several of our studies (Melhuish 2015; Melhuish & Fagan, 2017), Hazzan's original paper, and outside literature.

Table 1. Expanded Reducing Abstraction Framework

Abstraction Level as:	Operationalization of Reducing Abstraction
Relationship between the object of thought and the thinking person	• Moving from an unfamiliar concept/context to a familiar one[1] • Using familiar concept to bridge between unfamiliar concepts • Moving from decontextualized to familiar context
Reflection on the process-object duality	• Moving from an object to a algorithm • Moving from an object to a process[1]
Complexity of concept of thought	• Moving from a set to an element[1] • Moving from cohesive concept to disjoint parts • Moving from connected concepts to isolated concepts
Precision/formality of concept of thought	• Moving from formal definition to informal definition • Moving from definition to metaphor • Moving from formal definition to generic model

[1] Aligned with Hazzan's (1999) operationalization.

Relationship between the object of thought and the thinking person

Hazzan (1999) operationalized reducing abstraction in this sense via moving from an unfamiliar to familiar situation. She introduced an example where students engaged in tasks related to modular arithmetic groups and instead used properties and knowledge of familiar groups like the real numbers. This type of reducing abstraction occurs not when a particular concept is more general, but rather when it is new and unfamiliar. In many ways, a specific modular arithmetic group is just as concrete as a group like the reals. In our work, we similarly found students reverting to properties of familiar groups such as desiring identities to be either "0" or "1" regardless of binary operation.

We argue that Hazzan's (1999) second example, misapplying Lagrange's theorem to determine that $\mathbf{Z}_3$ is a subgroup of $\mathbf{Z}_6$ constitutes a parallel, but different way of reducing abstraction. In this application, a student does not replace an unfamiliar concept with a familiar concept, but rather uses a familiar concept (divisibility) as a bridge between the unfamiliar setting and an unfamiliar concept (Lagrange's Theorem). We found many students engaging in this type of abstraction reduction. For example, when students identified the size of cosets- rather than attend to the order of the subgroup used to build the coset- they provided the index as their answer. This illustrates a lack of familiarity with cosets bridged via a familiar concept divisibility to an unfamiliar concept, index.

In our third category, a student moves from unfamiliar (general) to familiar (specific) contexts. We saw students do this in a number of places in our data. For example, when students were asked to determine if the equation $(ab)^2=a^2b^2$ holds in groups generally, students returned to a number of familiar contexts including integers under multiplication (a misleading reduction of abstraction) or permutation groups (a productive reduction of abstraction). We see this activity as related to Gravemeijer and Doorman's (1999) referential activity in Realistic Mathematics Education designed tasks. During the process of reinvention of mathematical ideas, students often reduce abstraction and return to a specific context to productively explore ideas. (For an example, see Larsen and Lockwood's (2013) teacher-student exchange about left and right coset equivalence (p. 14).)

Process-Object Duality

As in the previous category, we subdivide Hazzan's (1999) process-object duality category. Process-object theories often distinguish between holistic processes and step-by-step actions (Tall et al., 1999). The use of "I" statements as highlighted by Hazzan (1999) may reflect algorithmic approaches where procedures are carried out step-by-step. Hazzan presented such an example where a student makes sense of the definition of a quotient group by explaining the canonical procedure for creating a coset using such language as "each one [element] by itself" (p. 81) as she walks through the relevant product creation.

We see this individual algorithm or action as one way to reduce abstraction. However, students may also go from an object and de-encapsulate (productively) to a process or inappropriate replace an object with a process (unproductive). For example, when a student was asked to find the kernel of a specific mapping, they responded, "The kernel of the homomorphism is what is inputted in $\boldsymbol{Z}$ [domain] to output the identity in $\boldsymbol{H}$ [codomain]." The student continued to treat the homomorphism holistically and identify the correct kernel set. This was a productive reduction in abstraction as de-encapsulating the kernel allowed the student to leverage the holistic process to correctly identify the kernel. In contrast, many students reduced abstraction to an action and provided incomplete kernel sets often identifying only one specific element that mapped to the identity of the codomain. In general, this type of abstraction reduction captures object-process duality with varying degrees of sophistication.

Complexity of the concept of thought

Hazzan (1999) provided one conception of abstraction within this category: using elements rather than a general set. We found Hazzan's (1999) classification useful and observed students engaging in similar reductions of abstraction. For example, consistent with Asiala et al. (1997), many students conflated the equivalence of left and right cosets with the commutativity of their individual elements. However, we also identified other ways students reduced abstraction by reducing complexity. We expand this category to include: *Moving from cohesive concept to disjoint parts* and *Moving from connected concepts to isolated concepts*. An example of the former category can be found in Melhuish and Fagan (2017). Students' engaging with tasks around binary operations reduced abstraction via attending to only one property. When asked if a given function (such as x^3) is a binary operation, majority of students focused on one property: closure. Reducing abstraction to this property is productive in traditional tasks where there are two inputs, however unproductive in a setting where not all functions are binary. This example illustrates that a student may reduce abstraction by attending to one aspect or property of a concept rather than the totality. The consequences of the reduction may be unintentional, especially if the students' concept image does not contain all relevant properties.

Alternately, a reduction of abstraction can occur when students lose relationships between other concepts that connect to the meaning of a concept at-hand. For example, when students were asked to find the inverse of c in the Cayley table below (table 2), many students identified c, treating a as the implicit identity element. Note that the identity element is not in the first row and column. When asked to explain their thinking on this task, such students did not attend to the role identity played in the concept of inverses. Rather, students explained inverse as, "[i]t's the opposite element of an element." In this way, their abstraction level is lowered via loss of an important connection to another concept: identity. We see these additional complexity theories as related to abstraction theories of properties and theories of connectedness. Students may reduce abstraction via attending to only a subset of properties or alternately losing important connections to additional concepts.

Table 2. Cayley Table defined on set {a,b,c}

	a	b	c
a	c	a	b
b	a	b	c
c	b	c	a

Degree of precision/formality

This category was not from Hazzan's (1999) framework; rather this additional category emerged to reflect theories of abstraction such as Davydov (1990) where abstraction level reflects the transition of mathematical object from informal/imprecise and to delineated and concrete. This process is sometimes equated to formalization in advanced mathematics. We identified three literature-based ways abstraction can be reduced from this theoretical lens.

Students may replace a formal definition with an informal definition. Lajoie and Mura (2000) identified this type of abstraction reduction when students engaged in tasks related to cyclic groups. We similarly found students leveraged an informal definition of cyclic when tasked with determining if particular groups were cyclic. Their definitions often relied on a generating action: "Start with the unit element and keep piling that onto itself" until a group is created. Reducing abstraction to this informal definition will be successful for finite group, but become problematic for the infinite cyclic group. See Melhuish (2018) for a discussion of how such an informal definition may be supportive for understanding the convention of powers in group theory.

Students may also use metaphors as a way to reduce abreaction. Rather than deal with a concept mathematically, they may leverage a metaphor such as an input-output machine for function (e.g. Zandieh, Ellis, & Rasmussen 2017). When identifying a specific homomorphism's kernel, a student used such a metaphor: "All the elements in the integers that would fit through the function and go to the identity on the codomain. Then start plugging things into that function." The function is treated as something that elements are fitted through. This particular reduction in abstraction was productive when attempting to identify the kernel of a given map.

A third variant is through creation of visual representations. As noted by Sfard (1991), abstraction can be reduced through returning to a visual image which is "more tangible, and encourag[ing] treating them almost as if they were material entities" (p. 6). In our work, we found this type of reduction of abstraction to be infrequent, but often productive when it occurred. One example can be seen in Figure 1. This student was asked to determine if cosets can always be formed from a given subgroup *H* and if so, what their size would be. The student drew a generic group partitioned into cosets to reason that this can always occur and the size would be the same as *H*. The group *G* was represented as a visual that can be reasoned from. This type of abstraction reduction aligns with Pinto and Tall's (2002) generic abstraction.

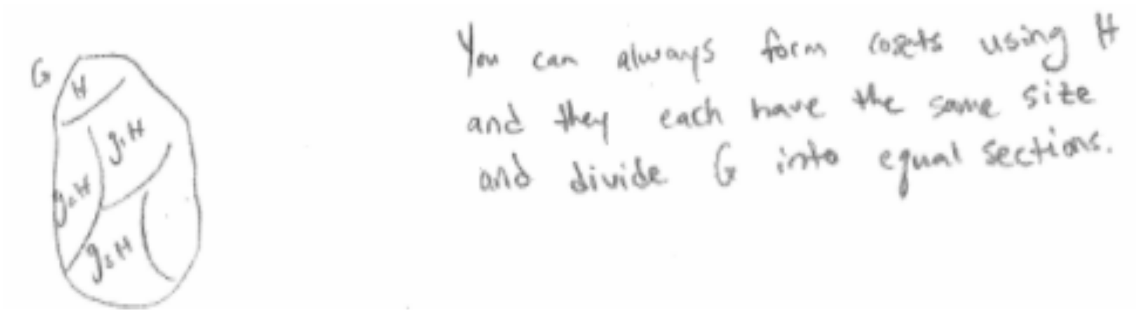

Figure 1. A student's visual representation of group G with cosets built using subgroup H

Discussion

In this paper, we sought to expand Hazzan's (1999) reducing abstraction framework by leveraging a number of abstraction theories. As Hazzan acknowledged, this type of framework cannot be exhaustive, nor mutually exclusive. In fact, the theories of abstraction which inform such an analysis often have overlap themselves. In this sense, we see this framework as a productive lens for analyzing student activity, but not a lens meant to categorize students. As Hazzan did originally, we made our theoretical expansions based on data from abstract algebra tasks. This subject area is populated with concepts that are abstract across many characterizations of abstraction (decontextualized, objects, complex, and stipulated.)

In addition to expanding the framework, we also wished to further highlight that reducing abstraction can be productive. Hazzan (1999) cautioned, "The term 'reducing abstraction' should *not* be conceived as a mental process which necessarily results in misconceptions or mathematical errors" (p. 75). However, Hazzan illustrated student activity that was either erroneous or neutral in problem-solving situations. While this if often the case, we also shared a number of examples of students working productively via reducing abstraction. In fact, we argue the ability to *appropriately* lower abstraction reflects a high level understanding. For example, to move from a formal representation to an accurate generic model reflects an advanced re-construction of a formal idea (von Glasersfeld, 1991). Similarly, Dubinsky and McDonald (2001) identified the ability to de-encapsulate from object to process as an essential feature of object-level conceptions. In this sense, we see parsing reduction of abstraction as more than just a tool for analyzing the cause of inaccurate student responses.

Such a framework can also provide insight into how we meet students where they are at in order to promote *productive* reduction of abstraction. There is power in being able to reduce abstraction in problem-solving (or proving) situations. Weber and Alcock (2004) presented contrasting cases where students (and graduate students) may produce proofs via working in an entirely formal system or through semantic explorations. In some sense, moving out of the formal system reduces abstraction level. It is this reduction that allowed successful provers in their study to gain insight into proofs. In Larsen and Lockwood (2013), students moved between decontextualized and contextualized situations to productively explore conjectures and ultimately reinvent mathematics. The question is not, how do we prevent students from reducing abstraction, but rather how do we promote students in reducing abstraction in productive ways? Through better understanding of reducing abstraction, we may ultimately aid in supporting students as they navigate abstract concepts in advanced mathematics.

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