Future Middle Grades Teachers' Solution Methods on Proportional Relationship Tasks

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This study examines the solution methods that future middle grades teachers chose when solving a problem on proportional relationships. The examination of the solution methods was framed by a new perspective on proportional reasoning that connects multiplication, division, and proportional relationships into a coherent framework. This framework places emphasis on multiple batches and variable parts. The data were collected from a sample of 22 future middle grades teachers' exams completed as part of a content course at a large university in the Southeastern United States. Findings revealed that future middle grades teachers utilized strategies involving multiple batches and variable parts after completing a two-semester sequence of mathematics content courses on proportional relationships tasks.

Keywords: Proportional relationship, Proportional reasoning, Variable parts perspective, Multiple batches perspective

Introduction

Skills of multiplicative and proportional reasoning are important because their development, or lack thereof, can greatly influence success for students in later mathematics (Beckmann & Izsák, 2015). First introduced in middle grades mathematics, reasoning proportionally forms a crucial base for further concepts such as functions, graphing, algebraic equations, and measurements (Karplus, Pulos, & Stage, 1983; Langrall & Swafford, 2000; Lobato & Ellis, 2010; Lobato, Orrill, Druken, & Jacobson, 2011; Thompson & Saldanha, 2003). Proportional reasoning is difficult, teachers are often not more advanced than their students and in order to teach effectively, one's own understanding must be deepened (Lobato et al., 2011). In addition, researchers pointed out that teachers need to be "sensitive to the types of reasoning that are most accessible as entry points for students while pushing them to develop more sophisticated forms of reasoning." (Lobato el al., 2011, p. 1). Despite the growing body of research on proportional reasoning, the studies that have explored future middle grades teachers' understandings of ratios and proportional relationships are rather limited. Thus, there is a need for research on how future middle grades teachers reason with proportional relationships because "teachers are among the most, if not the most, significant factors in children's learning and the linchpins in educational reforms of all kinds" (Cochran-Smith & Zeichner, 2009, p. 1).

Purpose of the Study

This study investigates the performances of future middle grades teachers in understanding proportional relationships from two distinct perspectives and considers the role of multiplication and division in their reasoning. Mathematics educators need new approaches and perspectives to think about how future middle grades teachers' reasoning about proportional relationships can be supported. With this objective in mind, Beckmann and Izsák (2015) developed a new approach to reasoning about proportional relationships comprising two perspectives and four methods. These methods encompass a coherent understanding of proportional relationships that includes both multiplication and division. Their approach was innovative because they connected multiplication, division, and proportional relationships into a single coherent framework that highlights two complementary perspectives on ratios and proportional relationships. These two perspectives are called variable parts and multiple batches. In line with Beckmann and Izsák's (2015) approach, this study specifically focused on future middle grades teachers' solution methods related to proportional relationships according to the two perspectives and four strategies. This research was guided by the question: What solution methods do future middle grades teachers use when solving a problem at the end of a content course involving two perspectives on proportional relationships?

Theoretical Framework

This study is framed by Beckmann and Izsák's (2015) perspective on proportional relationships, which integrates multiplication, division, and proportional relationships into a coherent whole.

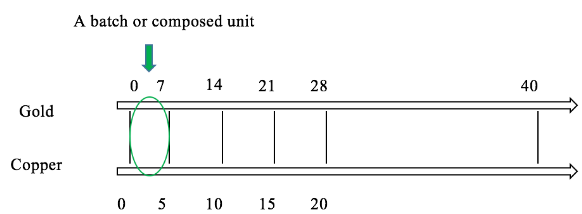
Equation: M•N= P

Beckmann and Izsák (2015) formalized an equation for multiplication based on equal sized groups as "M • N= P", where M is the number of the groups (multiplier), N is the number of the units (multiplicand) in each whole group, and P is the product amount.

Perspectives: Multiple Batches and Variable Parts

Beckmann and Izsák (2015) proposed two perspectives, multiple batches and variable parts, by considering the multiplier and multiplicand roles in proportional relationships. In this study, we demonstrate two perspectives by using the following Gold and Copper problem: *To make jewelry, jewelers often mix gold and copper in a 7 to 5 ratio. How much copper should a jeweler mix with 40 grams of gold?*

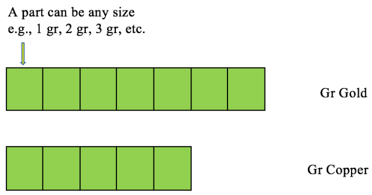
The multiple batches perspective supports at least two solution strategies: multiply-one-batch method and the multiply-unit-rate batch method. For the multiple batches perspective, they stated that "the original batch (A units of the first quantity and B units of the second quantity) are fixed multiplicands, and the multiplier varies; therefore, the proportional relationships can include "all of pairs (rA, rB)", where r > 0 (Beckmann, Izsák, & Olmez., 2015, p. 519). Figure 1 shows one way to represent multiple batches in the Gold and Copper problem.



Number of Batches	Gr Gold	Gr Copper
1	**1** • 7	**1** • 5
2	**2** • 7	**2** • 5
3	**3** • 7	**3** • 5
4	**4** • 7	**4** • 5

Figure 1: Multiple Batches Perspective (Beckmann et al., 2015)

Similarly, the variable-parts perspective supports at least two solution strategies: multiply-one-part method and the multiply-total-amount method. For the variable parts perspective, they considered the two quantities as consisting of A parts and B parts, respectively, where each part contains the same number of units. This time the multipliers are fixed by the numbers of parts, whereas the multiplicand varies with "the number of the measurement units" in every part (see Figure 2). Correspondingly, the multiple-batches perspective, variable-parts proportional relationships include "all of pairs (Ar, Br)" for r > 0 (Beckmann et al., 2015, p. 520). Figure 2 shows one way to represent variable parts in the Gold and Copper problem.



Gr per part	Gr Gold	Gr Copper
1	7 • **1**	5 • **1**
2	7 • **2**	5 • **2**
3	7 • **3**	5 • **3**
4	7 • **4**	5 • **4**

Figure 2: Variable Parts Perspective (Beckmann et al., 2015)

Methodology

Research Design

The aim of this study is to explore which solution methods future middle grades teachers used when solving a problem at the end of a content course that introduced two perspectives on proportional relationships. To address the research question, mixed methods were utilized to examine future teachers' solutions. Mixed methods research provides more evidence for studying a research problem than either quantitative or qualitative research alone. Quantitative methods individually provide useful information, however they do not provide an in-depth understanding of the participants approaches and qualitative research makes up for this weakness. Thus, the combination of strength of each approach accounts for the weakness of the other approach. More specifically, we used a sequential explanatory design with a qualitative approach being the first method applied as well as the method of priority (Creswell, Plano Clark, Gutmann & Hanson, 2003). Qualitative research methodologies were used to discover the meanings that participants created in context or in an activity (Wolcott, 2009). When reviewing the student written work qualitatively, we analyzed features of solutions and representations to determine the method they employed. In accordance with sequential explanatory design, it is typical to use qualitative results to reveal additional information and help clarify the primarily quantitative study (Creswell et al., 2003). Thus, we supported our qualitative interpretations with descriptive statistics. This combination of methods provides "multiple ways of seeing and hearing" (Greene, 2007, p. 20). With the priority placed on the qualitative approach, "the researcher builds a complex, holistic picture, analyzes words, reports detailed views of informants, and conducts the study in a natural setting" (Creswell, 2008, p. 15).

Data Collection

Data for the present study were collected from 22 future middle grades teachers at a large, public university in the Southeastern United States during the Spring 2016 semester of a two-semester mathematics content course. The first semester focused on numbers and operations including multiplication, division, and fractions; the second semester focused on topics related to fraction division, ratio, proportional relationships, and algebra. Both courses emphasized the meaning of multiplication. These courses were intended to help future teachers develop practices outlined in the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). The same textbook was used for both courses, *Mathematics for Elementary Teachers with Activities* (Beckmann, 2014). It was standard practice in these courses that future middle grades teachers worked in groups during class, however individually completed homework assignments and examinations.

Tasks on the midterm and final exams that addressed proportional relationships were identified. Then items that allowed future middle grades teachers to choose their own methods as opposed to items that directed them to use a particular method were chosen and analyzed. Ultimately, one task from the final exam of the second semester course was selected (see Figure 3).

Task	To make jewelry, jewelers often mix gold and copper in a 7 to 5 ratio. How much copper should a jeweler mix with 40 grams of gold? Write two different products A•B for the amount of the copper, where A and B are numbers derived from 7, 5, and 40. Explain each product A•B in detail in terms of the situation using our definition of multiplication and using math drawings as support.

Figure 3. Task Item

Data Analysis

Drawing on the theoretical framework, we were able to classify the future teachers solutions. This framework is exemplified in Figure 4, which shows solutions for the Gold and Copper problem that illustrate the two perspectives and four methods and how those methods are coordinated with equations following the multiplier (M) • multiplicand (N) convention. Beckmann and Izsák (2015) indicated that double number lines (DNLs) fit well with the multiple-batches perspective and that the strip diagrams fit well with the variable-parts perspective. DNLs represent quantities visually as lengths and afford such operations as iterating, partitioning, or addition. Strip diagrams represent quantities in terms of variable parts.

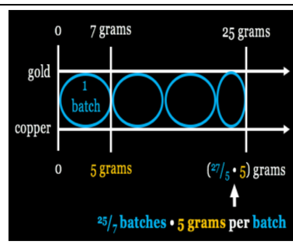
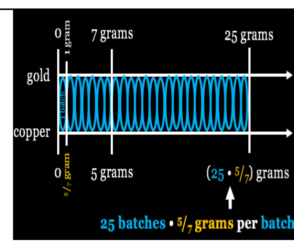
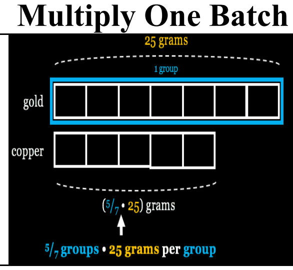
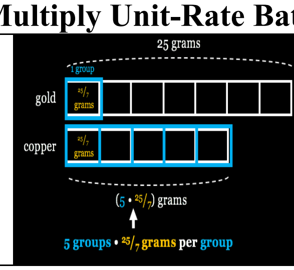
<table>
<tr><td colspan="3">Gold and copper problem: “A company makes jewelry gold using gold and copper. The company uses different weights of gold and copper on different days, but always in the same ratio of 7 to 5. If the company uses 25 grams of gold on one day, how much copper will they use?”</td></tr>
<tr><td>Multiple Batches</td><td>
</td><td>
</td></tr>
<tr><td>Strategy</td><td>Multiply One Batch</td><td>Multiply Unit-Rate Batch</td></tr>
<tr><td>Variable Parts</td><td>
</td><td>
</td></tr>
<tr><td>Strategy</td><td>Multiply Total Amount</td><td>Multiply One Part</td></tr>
</table>

Figure 4. Solutions to the Gold and Copper Problem using two perspectives on proportional relationships and four strategies (Reproduced Kulow, 2017)

The data were sorted based on the perspective future teachers chose (multiple batches or variable parts) and then based on methods that fit with those two perspectives. Task analysis focused on the future teachers’ solutions according to their drawings, equations, and explanations.

Results

Future middle grade teachers who completed the two-semester sequence of content courses emphasizing topics related to ratio, proportional relationships, fraction division, algebra, and the meaning of multiplication were able to appropriately use the multiple-batches and variable parts perspectives. When working on a problem that allowed them to select their own method, future middle grades teachers tended to use the variable-parts perspective as opposed to the multiple batches perspective.

Table 1 shows counts for solution classifications to the Gold and Copper problem. Recall that the task asked for two solutions. The counts in Table 1 show that 44 solutions were provided by 22 future teachers: 19 future teachers used two different methods, two future teachers used one method, and one future teacher used four methods, as shown in Table 1. According to these results, the future teachers used the variable-parts perspective in 29 solutions and the multiple batches perspective in 15 solutions.

Table 1. Frequency of each method

Perspective	Total	Method	Total
Variable Parts Perspective	29	Multiply Total Amount	12
		Multiply One Part	17
Multiple Batches Perspective	15	Multiply One Batch	8
		Multiply Unit-Rate Batch	7
	Total		44

The total number of solutions in which future teachers used the variable parts perspective with the multiply-total-amount method was 12, whereas the total number of solutions in which future teachers used the variable parts perspective with multiply-one-part method was 17. Additionally, the total number of solutions in which the future teachers used the multiple batches perspective with multiply one batch method was 8. The total number of the solutions in which future teachers used the multiple-batches perspective with multiply unit-rate batch method was 7. Some future teachers used the multiple batches perspective with multiply-one-batch method logically in combination with a strip diagram instead of a DNL. Some future teachers used the multiple batches perspective with multiply-unit-rate-batch method logically in combination with a strip diagram instead of a DNL.

Variable-Parts Perspective with the Multiply Total Amount Method

Future teachers who used the variable parts perspective with the multiply-total-amount method included an equation which mainly included appropriate values for M and N (i.e., M is 5/7 and N is 40). For instance, the future teacher LM defined M = 5/7 as “# of groups”, N = 40 as “# of grams in one whole group”, and P = 200/7 is “# grams in 5/7 group”. In addition, LM showed the total amount of gold and copper in the math drawing (see Figure 5).

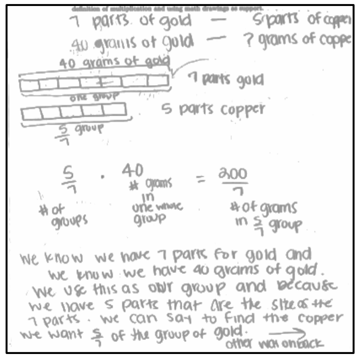


Figure 5. LM's solution

Variable-Parts Perspective with the Multiply One Part Method

Future teachers who used the variable parts perspective with the multiply one-part method included an equation with appropriate values for M and N (i.e., M = 5, N = 40/7, and P = 200/7). The future teacher BM stated M is "# of groups", N is "units per group", and P is "amount of copper needed" (Figure 6).

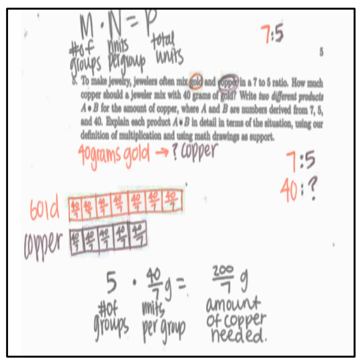


Figure 6. BM's solution

Multiple-Batches Perspective with the Multiply One Batch Method

Future teachers who used the multiple-batches perspective with the multiply one batch method included an equation which mainly included appropriate values for M and N (i.e., M = 40/7, N = 5, and P = 200/7). Figure 7 includes future teacher AH's solution using the one batch method that included explicit descriptions for M, N, and P such as M is "groups gold", N is "grams copper per group", and P is "grams copper per 40 grams gold."

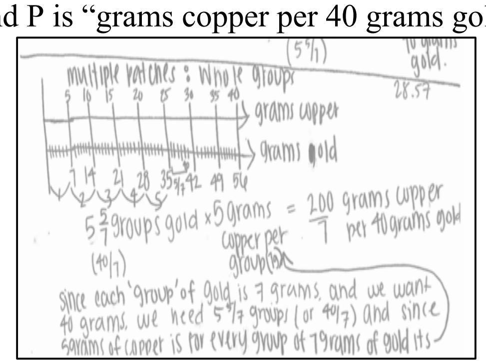


Figure 7. AH's solution

Multiple-Batches Perspective with the Multiply Unit-Rate Batch Method

Future teachers who used the multiple-batches perspective with the multiply unit-rate batch method included an equation which mainly included appropriate values for M and N (i.e., M is 40, N is 5/7, and P is 200/7). In Figure 8, future teacher KC used the mathematical drawing, showed total amount of gold and copper. More specifically, in KC's solution, DNL indicated target amount (e.g., tick mark for 40 grams of gold) and DNL indicated initial batch (e.g., tick mark for 7 grams of gold and 5 grams of copper) (see Figure 8).

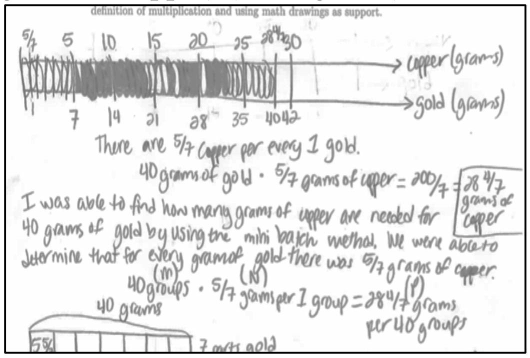


Figure 8. KC's solution

Discussion and Conclusion

Proportional relationships are at the heart of middle grades mathematics, so learning and teaching this concept is crucial. In order to improve learning the concept of proportional relationships, we need to educate future teachers. Thus, there is a need for research on the mathematical training of future middle-grade teachers for better teaching and learning of proportional relationships between co-varying quantities. In order to reach this goal, the education program for future middle grades teachers should be designed to support proportional reasoning. The findings of this study indicated that when topics related to ratio, proportional relationships, fraction division, algebra, and the meaning of multiplication were emphasized in a two-sequence content course, future middle grades teachers were able to use the multiple-batches and variable-parts perspectives and the associated methods in an appropriate way on an exam problem.

This study revealed that two perspectives are important since both have been designed by combining multiplication, division, and proportional relationships. While the sample size of the study is small, more participants are needed in more classes for future work. In addition, studies including interviews are needed to further understand future teachers' solution methods by considering two perspectives.

The instructional approach to topics in the multiplicative conceptual field appeared to support development of future middle grades teachers' understanding of proportional relationships. This approach also supports future teachers' understanding of the meaning of multiplication and division and the use of each perspective's features. According to Beckmann and Izsák (2015), the variable-parts perspective offers students an approach to thinking about variations of quantities in proportional relationship problems. In this study, students used the variable-parts perspective (n = 29) more often than the multiple-batches perspective (n = 15). This result represents the first determination regarding students' tendency when choosing which perspective to work with.

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