Integrals, Volumes, and Visualizations

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Many studies have been done on student understanding of integration and this research aims to add to that knowledge base with the study of student understanding of integration when applied to volume problems and how visualizations and sketches are used in the problem-solving process. Participants were recruited from a large, public, research university and interviews consisted of students working through routine and novel volume problems while discussing their thought processes aloud. Preliminary results show that students rely heavily on memorized formulas and have difficulties explaining the concepts behind the formulas. The idea of the integral as a sum of small pieces is present in most students studied, but they have trouble relating this idea to the formulas in their volume integrals. All students drew sketches of the geometric situation for all the problems, but the extent to which they could use their sketch meaningfully varied greatly.

Keywords: Calculus, definite integral, volume, visualization, Riemann sum

Introduction and Literature Review

After an introduction to the concept of the definite integral, some of the first applications that students encounter are volume problems. Volume problems found in second-semester calculus classes involve a combination of visualization, geometry, and integration skills. Previous studies have found that when solving definite integral application problems, students often rely on formulas, patterns, and previously encountered methods for setting up integrals (Yeatts & Hundhausen, 1992; Grundmeier, Hansen, & Sousa, 2006; Huang, 2010). In one of the first studies on student understanding of integration, Orton (1983) found that students had very little idea of the dissecting, summing, and limiting processes involved in integration when solving area and volume problems. Several authors (Sealey, 2006, 2014; Jones, 2013, 2015a, 2015b; Meredith & Marrongelle, 2008) have found that students are most successful when they are able to conceptualize the definite integral as the limit of a sum of products. Moreover, Sealey's (2006, 2014) work shows that students may have an idea of the underlying structure of the definite integral, but may not fully understand the layers that comprise the whole. In particular, students can easily conceptualize the summation layer but have the most trouble when working in the product layer of the Riemann sum structure.

One key component of a calculus volume problem that students can use as an aid is a visualization of the situation, generally in the form of a sketch made by the student. Stylianou and Silver (2004) found that, even though the construction of a diagram or picture is helpful, it is the quality of the picture that is most important. Bremigan (2005) had similar results, finding that although diagram production was related to correctly solving the problem, the presence of a constructed or modified diagram was not a sufficient condition for problem-solving success. In their study on expert and novice visualization practices, Stylianou and Silver (2004) observed novices' cognitive disconnect between visualizations and the problem-solving process. They state that, "although novices appear to have aspects of the declarative knowledge associated with visual representation use, they lack the necessary procedural knowledge that would allow them to use visual representations functionally and efficiently" (p. 380).

Research Aim

As volume problems are one of the first applications of the definite integral that students encounter, the aim of this study is to further explore how students view and use the underlying structure of the definite integral when solving these types of problems. We are also interested in how students use their sketches of the geometric situation to aid in solving volume problems.

Conceptual Framework

Sealey's (2014) Riemann Integral Framework was used to inform both the data collection and analysis of student understanding of the structure of the definite integral. This framework breaks the constituent parts of the Riemann integral down into pieces – product, summation, limit, and function – and it allows us to pinpoint the parts of the underlying structure of the definite integral that students have the most trouble with when solving volume problems. For the visualization aspect, we will be using Zazkis, Dubinsky, and Dautermann's (1996) Visualization/Analysis Framework to analyze student use of pictures and diagrams in the volume problem-solving process. In this model, there is a first visualization, V_1 (for example, a sketch of a 2-dimensional region), which is then acted on by an analysis event, A_1. In the following act of visualization, V_2, the student is still attending to the same picture used in V_1, but its nature has changed (due to A_1) and could lead to a reinterpretation of the picture or a new image construction. No matter the form the visualization takes in this step, V_2 results in a richer understanding of the original situation. The process goes on like this, from visualization to analysis back to visualization, optimally resulting in a more complete understanding of the physical situation.

Research Methodology

Interviews with students were conducted during summer 2016 (Study 1) and summer 2017 (Study 2). The participants were recruited from summer classes at a large, public, research university. In summer 2016, the participants were four Calculus 2 students (all male) and three Elementary Differential Equations students (one female and two male). In summer 2017, the participants were two Calculus 2 students (one male and one female). The interviews were one-on-one and videotaped, and the students were asked to write their math work on paper or a white board and discuss their thoughts aloud.

During the interviews, the students were asked to complete three second-semester integral volume problems. In Study 1, the problems were three routine solid of revolution problems (e.g., "Find the volume of the solid obtained by rotating the region bounded by the curves $y = x^2$ and $y = 3x$ about the line $x = -1$") and the students drew their sketches and wrote their math work on the same paper. In Study 2, the problems were two routine solid of revolution problems and one geometric solid problem, which we will call the pyramid problem ("Find the volume of the pyramid whose base is a square with side length L and whose height is h."). In order to more clearly observe when students were referring to their drawings during the problem-solving process, the method was adjusted in Study 2 as follows. One student in Study 1 drew sketches and performed math work on separate sheets of paper; the second student sketched drawings on a white board and wrote math work on a sheet of paper.

During the interviews for both studies, students were probed about their responses and were asked to explain their work and thought processes. Some typical questions asked during the interview process were: "How do you know this integral gives you a volume?", "How does that particular statement give you the volume of a cylinder/washer/etc?", "What does the *dx* mean?", and "Can you show on your picture the different parts of the volume integral?"

The video data was transcribed and analysis is in the beginning stages. We use thematic analysis (Braun & Clark, 2006) to identify themes and patterns in the data. In particular, we have begun by employing theoretical thematic analysis, which is "driven by the researcher's theoretical or analytic interest in the area, and is thus more explicitly analyst-driven" (p. 84). We feel that this will be an appropriate method for this study, since we have pre-determined codes that we will be looking for in student responses (e.g., working in a specific layer for Sealey's framework or being in one of the visualization/analysis phases of Zazkis' framework).

Preliminary Results

The students in these studies exhibited a strong attachment to memorized volume integral formulas when solving the routine solid of revolution problems.

Interviewer: Do you understand where that [*their volume integral*] came from?
Student 2: I treat that just as a formula. Physics is the class where I think about and understand, you know, but, it just could be because they throw a lot of numbers at you fast.
Student 3: I know the formula, but sometimes I don't know where to apply them.

Using the formulas is not a detrimental method, but we would prefer that students are also able to unpack the underlying definite integral structure when asked to do so. Few students in this study were able to accurately and consistently discuss the details of their memorized formulas or how they produced a volume measurement.

Another observed occurrence was students linking the line of rotation to the variable of integration without being able to produce meaningful explanations.

Student 1: Since the line I'm rotating about is parallel to the y-axis, if I use cylindrical shells method, I need to integrate x.
Student 2: So, if, like it's [*the line of rotation*] parallel to the y-axis, I'll integrate y.

There were only two students who were given the pyramid problem, but their methods for attacking the problem were very different and highlighted some problems of relying on memorized formulas and mimicking methods seen in class. Student 5 was very successful with the routine solid of revolution problems by relying heavily on the "volume formulas" and was able to produce accurate volume integrals that would receive high marks on an exam, even though her explanation of the details was shaky. When confronted with the pyramid problem, Student 5 continued to try to use a memorized volume formula but with poor results. Student 5 had to be heavily guided in the pyramid problem due to over-reliance on memorized formulas that did not fit with this problem. Even though Student 5 did not succeed in solving the problem completely, there was the presence of "cutting into small pieces and adding them up" in her thought process that seemed to be accessible but not heavily used. Student 5 also had a very hard time visualizing the situation and had to be heavily guided into a 2-dimensional side-view of the pyramid so that she could reduce the cognitive load of the 3-dimensional solid situation.

Student 6 was less successful with the solid of revolution problems; he relied heavily on the memorized formulas but was unsure of how to use them effectively. When probed using the questions stated above, he was very unsure and stated this fact many times. When confronted with the pyramid problem, he had trouble visualizing the situation at first, but then transformed the 3-dimensional solid into a 2-dimensional side view (on his own) and was able to make significant progress. Once the weight of "you must use this formula" was lifted, Student 6 was able to make some strong connections between what he knew of "adding up small pieces" and this novel volume problem. This problem forced Student 6 to give up on his attachment to memorized volume formulas and rely solely on the concept of the definite integral as the sum of smaller pieces.

The students in this study had many misunderstandings but we want to emphasize more what they could do than what they could not do. When faced with volume integral application problems, almost all of the students exhibited some understanding of the dissecting, summing, and limiting structure of the definite integral; they just had trouble applying it to the problem.

> *Student 1*: In that case you're using the, um, areas of [*stacks hands horizontally on top of each other*] … circles. Um, so you're making a series of washers.
>
> *Student 1*: So if you were to take one of those, a slice of the inside of the cylinder, it would be like this sheet with a depth. But when we integrate, we're basically taking that depth to zero. The limit of that depth. Right?
>
> *Student 2*: What I'm looking for then is my radii of my, you know, infinitely concentric circles going on here.
>
> *Student 4*: Basically it's going to be a bunch of different-sized cylinders stacked upon each other.

Students' drawings varied in sophistication and accuracy. All students drew a sketch of the geometric situation for all the problems, but the extent to which they could use their sketch meaningfully varied greatly. The most glaring disconnects came when students produced a correct (or mostly correct) volume integral, but could not relate their integral back to the physical situation of the sketch. One student in particular – Student 2 – was able to produce very detailed sketches of the sum-of-pieces structure (Figure 1), but his description of the underlying mathematics was full of inaccuracies and nonsense. According to Sealey's (2014) framework, this student is appropriately attending to summation layer, but he had great difficulties in the product layer.

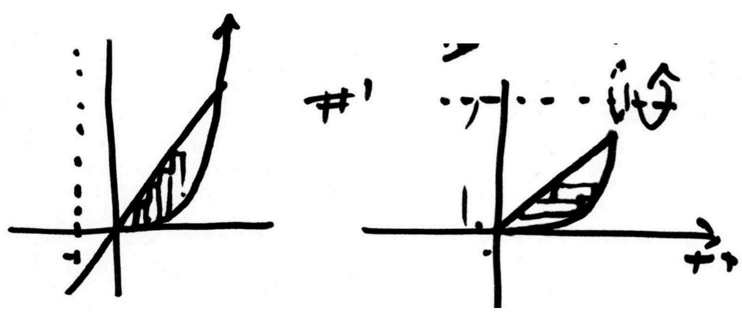

Figure 1: Drawings produced by Student 2 when describing his volume integrals.

Teaching Implications/Future Research

The preliminary results of this study imply that students' heavy reliance on memorized formulas and mimicking of methods observed in class can lead to misunderstandings and brick walls when students are faced with more complicated or non-routine integral application problems. It was also observed that producing an accurate definite integral to a solid-by-revolution problem does not necessarily imply that the student understands integration and how the integral produces a volume. Solid by revolution problems are given a lot of face time in most calculus books, but they can inadvertently tell an incomplete story of how integrals can be used to find volumes in general and how they can be used in other application situations. We believe that more time should be spent on non-revolution and non-routine volume problems, so that students are required to exercise their definite integral muscles and not be tempted to fall into the trap of relying solely on memorized formulas. From this study, it is clear that the idea of "dissecting and summing" is present in many students, and we need to find ways to employ and enrich it.

We plan to continue analyzing our data set, as well as conduct more interviews with the aim of examining student visualization and picture-use when solving volume problems, like the interviews conducted during summer 2017 involving the pyramid problem. In particular, we would like to investigate how they use their sketches to build the pieces of the corresponding volume integral and how they interact with their drawing in the problem-solving process. Furthermore, we would like to develop ways in which students can more meaningfully engage in constructing and understanding the product layer. In the future, we would like to study other aspects of visualization that students use when solving volume problems, like gesture.

Questions for Audience

1. Aside from eye-tracking software, are there more sophisticated ways to capture when students are going back and forth between picture and math work on camera?
2. How can we determine if separating the drawing space from the math work space bring about any unintended consequences for student problem-solving?
3. Are there any studies on visualization that we have missed that could help us out with this work?

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