How Does Problem Context Shape Students' Mathematical Reasoning on Calculus Accumulation Tasks?

William Hall
Washington State University

Calculus serves many students from myriad fields of study. Investigations into the ways students from these fields of study reason about calculus concepts are vital, yet lacking (Rasmussen, Marrongelle, & Borba, 2014). The biological and life sciences make up 30% of traditional Calculus I students (Bressoud, 2015) and yet we know very little about how these students utilize context as they reason about calculus ideas like the definite integral. In this study, task-based interviews were conducted with 12 undergraduate students majoring in the biological and life sciences. Data were analyzed via open coding from a constructivist grounded theory approach (Charmaz, 2000) and a new analytic tool, local theory diagrams was developed. Results indicate problem context influenced students' assessment of the viability of their solution strategies as well as enabled them to reason through apparent contradictions in their work.

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Framing the Study

Calculus is at the heart of a great many disciplines. Biology, computer science, economics, engineering, and physics are just a few of the undergraduate programs that require at least one semester of calculus. Enrollment in calculus courses at the secondary and post-secondary levels continues to rise (Bressoud, Carlson, Mesa, & Rasmussen, 2013; Kaput, 1997) and so understanding how students reason about calculus concepts is vital to better serve this growing community. Since the 1980s, research in calculus teaching and learning has blossomed into a field unto itself where researchers have explored several areas including the cognitive development of introductory calculus concepts in students and the potential for new digital tools to change calculus instruction (see Rasmussen, Marrongelle, & Borba, 2014 for a review).

Recent studies have highlighted the service nature of introductory calculus at the undergraduate level, since "very few students in Calculus I - between 1% and 3% of those enrolled in this course - intend to major in mathematics" (Bressoud et al., 2013, p. 691). Most students in these classes are majoring in other fields, what are often called the *client disciplines* of calculus. One popular client discipline of calculus is the biological and life sciences. Researchers have identified that 30% of the students in traditional Calculus I courses intend for careers in the biological and life sciences (Bressoud, 2015). However, the traditional Calculus I course "is designed to prepare students for the study of engineering or the mathematical or physical sciences" (Bressoud et al., 2013, p. 691). Which means a great many students in calculus are not seeing many contextually-based tasks catered to their field of study.

This study specifically addresses students' solution strategies on tasks involving the definite integral and accumulation primarily because integration and accumulation serve an important role in differential equations, which are used extensively in modeling within the biological and life sciences. Researchers have investigated student conceptions of the definite integral and have found that calculus students are good at using the standard antiderivative techniques taught in introductory calculus (Ferrini-Mundy & Graham, 1994; Grundmeier, Hansen, & Sousa, 2006; Mahir, 2009; Orton, 1983) and that while area under the curve dominates instruction of the definite integral in calculus, the multiplicative structure of the Riemann sum is a more powerful way to conceive of the definite integral as seen in both mathematics and physics education research (e.g. Jones, 2015a; Sealey, 2014). Unfortunately, researchers have seen that students struggle to make these meaningful connections between rate of change and accumulation in definite integral tasks (Bajrachara & Thompson, 2014; Thompson, 1994). Furthermore, researchers have found that when solving physics-based tasks, students' problem-solving strategies differ in relation to the context presented (e.g., Bajracharya & Thompson, 2014; Jones, 2015b; Sealey, 2014), and that some of these strategies are productive in a physics context when compared to a decontextualized mathematics context (Bajracharya, Wemyss, & Thompson, 2012; Jones, 2015a).

To better serve students from the myriad client disciplines of calculus, we must understand how students solve calculus tasks set in contexts relevant to those fields and whether those contexts play a significant role in their mathematical reasoning. Rasmussen et al. (2014) end their review of the state of research on calculus teaching and learning with a call for "research that closely examines the ways in which calculus ideas are leveraged in the client disciplines, how these ideas are conceptualized and represented in the client disciplines, and what these insights might mean for calculus instruction" (p. 513). The current study was designed to address this gap in the literature. My specific research question is: What role does context play in how undergraduate students majoring in the biological and life sciences solve calculus tasks involving accumulation?

Theoretical Perspective

The perspective of learning that influenced the construction and analysis of this study is constructivism, specifically a view of knowledge as cognitive adaptation. In a constructivist theory of learning, the fundamental assumption is that learners build up knowledge for themselves instead of being imbued with knowledge by those around them. In other words, the learner must actively participate in the development and organization of the cognitive structures making up their understanding of the world (von Glasersfeld, 1982). To explore an individual's understanding, one must consider the following three factors: "the individual's current state of development, social and cultural influences of a tribe (group), and environmental/physical factors in relation to the task at hand" (Confrey & Kazak, 2006, p. 317). This perspective on learning, while maintaining focus on the individual learner, acknowledges that social and environmental factors must necessarily play a role in that learning. For this study, such a perspective provides the foundation for analyzing individual's approaches to calculus tasks while framing those approaches within the influence of those individual's backgrounds (in this case, as undergraduate students majoring in the biological and life sciences) and the interview setting itself.

One aspect of constructivism that played a key role in the data analysis in this study is a view of knowledge as an adaptive function. Ernst von Glasersfeld, in his interpretation and extension of the work of Jean Piaget, stresses the connection between the mechanisms of evolution by natural selection and how individuals learn. von Glasersfeld (1982) claims "knowledge for Piaget is never (and can never be) a 'representation' of the real world. Instead it is the collection of conceptual structures that turn out to be adapted, or as I would say, viable within the knowing subject's range of experiences" (p. 4). Viability is the crucial idea. Just as with the evolution of an organism in an ecosystem, what students learn is not driven by matching some objectively true reality, but what the student, within their personal "ecosystem," finds viable. Therefore, for learning, as in evolution, there is an emphasis placed on stability and equilibrium. von Glasersfeld states that "in the sphere of cognition, though indirectly linked to survival, equilibrium refers to a state in which an epistemic agent's cognitive structures have yielded and continue to yield expected results, without bringing to the surface conceptual conflicts or contradictions (p. 5). This is the heart of the concept of viability in constructivism, that learning is the development of stable cognitive structures and forms the foundation for the analytical tool developed herein, local theory diagrams, which were designed to highlight this process of students assessing the viability of their mental schemes.

Methods

To answer the research question posed, qualitative methods were employed. I utilized task-based interviews with twelve undergraduate students majoring in the biological and life sciences at a large public university in the Southeastern United States that I will call South State University (SSU) in the spring of 2016. Task-based interviews allowed me to investigate students reasoning about calculus tasks involving accumulation and to probe their understanding as they solved the problems. Data were open-coded via methods from constructivist grounded theory (Charmaz, 2000) which led to the development of a new analytic tool, local theory diagrams.

Participants

The population was all undergraduate students majoring in the biological and life sciences at South State University (SSU). SSU is a large, public university serving approximately 24,000 undergraduates. The students at SSU are of high academic caliber; half of all incoming freshman rank in the top ten percent of their high school class with a GPA of at least 3.75. SSU is considered “very selective” with 46% of applications admitted per year (The College Board, 2017). Students majoring in the biological and life sciences at SSU at the time of this study, were required to take at least two semesters of calculus, either the calculus sequence for life and management sciences or Calculus I and II.

Participants were solicited by visiting second semester calculus courses specifically designed for students studying in the biological and life sciences as well as upper-level courses within the biological and life sciences. Twelve students were interviewed, half of which were freshman or sophomores while the other half were juniors or seniors. The students were predominately female (8 of 12) and Caucasian (11 of 12).

Interview Protocol

In this study, I utilized task-based interviews in which students completed five calculus tasks concerning accumulation (approximately 50 minutes). In each of the five tasks, the students were presented with a rate of change function of some quantity and asked questions about the accumulation of said quantity over various periods of time. To answer the research question: “What role does context play in how undergraduate students majoring in the biological and life sciences solve calculus tasks involving accumulation?”, the contexts for the tasks were chosen to be diverse but relevant for the students’ backgrounds. In this session, I will discuss the results of two of the tasks, which are reproduced below in Figures 1 and 2.

Task 3

The greenhouse effect is the rise in temperature that the Earth experiences because certain gases prevent heat from escaping the atmosphere.

According to one study, the temperature is rising at the rate of: $R(t) = 0.014t^{0.4}$ degrees Fahrenheit per year, where t is the number of years since 2000. Given that the average surface temperature of the earth was 57.8 degrees Fahrenheit in 2000, predict the temperature in 2200.

Figure 1. *Interview Task 3*

Task 5

A new virus has shown up in a small town. The following table gives the estimated rate of change of the number of infected individuals on a given day (measured in people per day). Describe how the number of infected people is changing and use the information in the table to estimate the total change in the infected population over the given time period.

Days	0	2	4	6	8	10	12	14	16
Rate (Infected per day)	18	25	20	4	-5	-10	16	17	20

Figure 2. Interview Task 5

Data Analysis

Analysis of the interview transcripts followed a constructivist grounded theory approach (Charmaz, 2000). Constructivist grounded theory, like other forms of grounded theory (e.g., Glaser & Strauss, 1967; Strauss & Corbin, 1990), allows the researcher to explore the data without a preconceived framework of what results should emerge from the data. Charmaz notes that objectivist grounded theorists "assume that following a systematic set of methods leads them to discover reality and to construct a provisionally true, testable, and ultimately verifiable 'theory' of it" (p. 524) and therefore that the data collection and analysis procedures should aim to minimize the role of the researcher to be able to make claims about an observer-independent reality. For constructivist grounded theory, Charmaz argues, this is not the case. She argues that "the research products do not constitute the reality of the respondents' reality. Rather, each is a rendering, one interpretation among multiple interpretations, of a shared or individual reality" (Charmaz, 2000, p. 523). Charmaz illustrates this succinctly when she says, "data do not provide a window on reality. Rather, the 'discovered' reality arises from the interactive process and its temporal, cultural, and structural contexts" (p. 523-524). This approach to data analysis fits with the theory of learning described earlier, particularly the focus on viability in learning since both perspectives reject the assumption that we are uncovering some objectively true reality.

The open-coding process led to the development of a new analytical tool, local theory diagrams, which visually represent a student's solution strategy and all its mutations for a given task. Local theory diagrams showcase the "core" of the student's current theory concerning the given task and its solution (e.g. how to interpret the given rate of change function) and is then surrounded by all the hypotheses the student generates based on that assumption and ideas the student believes to be true at the time. The local theory diagrams also illustrate how these theories shift as the student interacts with the task and assesses whether their current assumptions and strategies make sense. In this way, the diagrams show the process of students coming to develop more viable theories of the tasks they solve. Examples of the local theory diagrams are given in the next section.

Results

There were two primary ways the problem context helped shape students' mathematical reasoning. The first was their use of the context partnered with the given information to refine their local theories of the task to increase the perceived viability of their strategies. Secondly, students would occasionally use the problem context to help explain away apparent contradictions within one of their local theories. I will use the results of the open-coding process as well as a few examples of the local theory diagrams to illustrate each of these findings.

Using Problem Context and Given Information in Theory Refinement

Whenever the students began working through one of the accumulation tasks, they were continuously revising or replacing a local theory concerning the task. For Task 3, there were a few pieces of information students attached to while generating various solution strategies. Primarily, students knew that because the initial temperature was given to be 57.8 degrees Fahrenheit and the problem concerned climate change and the warming Earth, that their answer must be greater than 57.8 degrees Fahrenheit. Seven of the 12 students interviewed initially assumed that the given function would output the average surface temperature in the year 2200. This assumption runs contrary to the actual problem text in which it is stated that "the temperature is rising at the rate of: $R(t) = 0.014t^{0.4}$ degrees Fahrenheit per year." While many of them read the task out loud prior to beginning their work, they neglected this specific description of the function as a rate and instead assumed it represented the average surface temperature.

With this assumption, each of the seven students then evaluated $R(200)$ and were then faced with contradictory evidence since $R(200)$ equals approximately 0.116. Each of the seven students then realized their current theory was no longer viable, their understanding of what the answer to the task should be overwrote their assumption that the function would output the average surface temperature and so a new local theory was developed to explain this new contradictory evidence. As Tom acknowledged after seeing the result of $R(200)$, "and I said that was wrong because I was, wait, that's so small." It is important to note that this realization does not necessarily lead the students to interpret the output of the function as it was intended, as a rate of change. When contradictory evidence is acknowledged, the student adjusts their local theory or abandons it for another local theory that is more viable to them. In five of the seven interviews in which students acknowledged this contradictory evidence, the student then developed a second local theory with the core assumption that the function is outputting the change in the average surface temperature instead of the average surface temperature itself. Thus, the students tend to suggest adding 0.116 to 57.8 to find the new average surface temperature. This new hypothesis is more viable for the students since it fits within the contextual assumptions they have made. This hypothesis is not mathematically accurate. The students are adding a value of the instantaneous rate of change to the initial temperature instead of using the rate of change to approximate or calculate the change in the temperature over the 200 years. However, the students do not tend to perceive any contradiction here, their current local theory is viable to them since the contextually-based assumptions are now not in any perceived contradiction with the evidence. The fact that their solution is mathematically inaccurate is not a factor in the students' assessment of viability.

In Figure 4 below, we see such an example in Anne's local theory diagram for her work on Task 3. We see in Anne's first local theory that her core assumption is $R(t)$ outputs the average surface temperature. While she describes the function as the "rate of change for the temperature" she then claims that by plugging in 200 into the equation she will get the average surface temperature in the year 2200. After calculating $R(200)$ she notes that this is a very small value and after I ask her what the function tells her about the context she drops her current theory for a more viable one, one with a core assumption that the function outputs the change in the average surface temperature. This theory is more viable for Anne since she is now able to explain her formerly contradictory information that $R(200)$ is a small number. I ask Anne specifically what the units on $R(t)$ are with the intention of seeing if this will cause her to acknowledge another contradiction but she is content in stating the units are degrees Fahrenheit per year without acknowledging any contradiction in her theory. Later in the interview, after she had completed all the other tasks, I again direct her attention to the units and ask her if she can add degrees per year to degrees. Now, based on this question and my desire to return to this task, Anne shifts to another local theory with the core assumption that the function outputs the rate of change of the average surface temperature instead of the actual temperature or the temperature change. Anne now reasons that the rate of change would vary each year and so she would have to add the value each year to the starting temperature of 57.8 degrees. Anne has now utilized the context of the task, the given information from the task, and the interview setting to continually revise her local theory about the task and so her final local theory would be considered the most viable for her at the end of the interview.

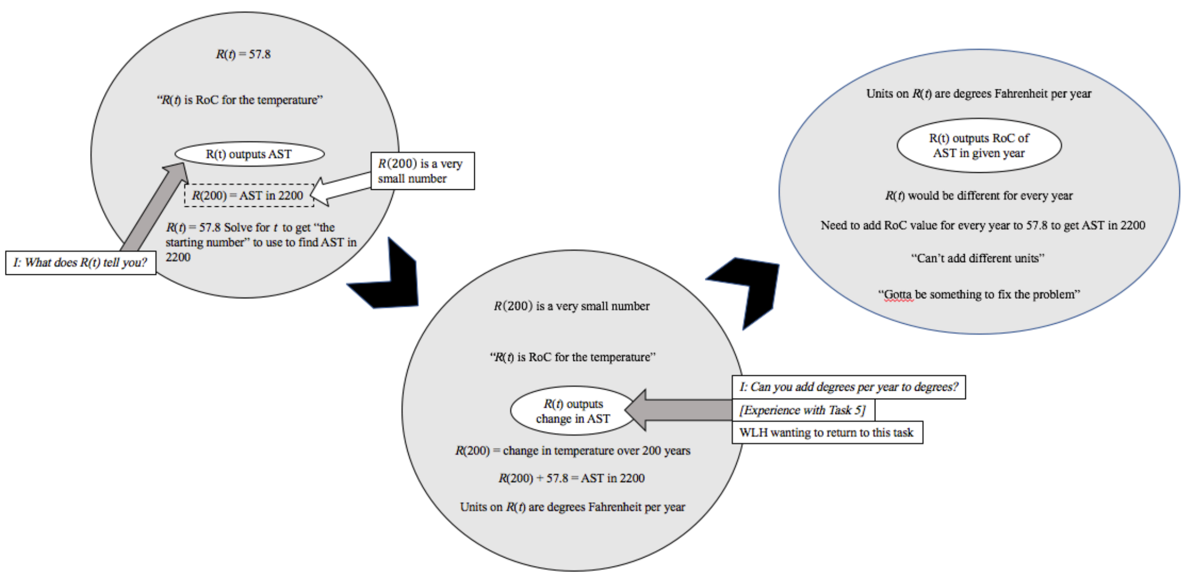

Figure 3. Local theory diagram for Anne's work on Task 3

Reasoning About Contradictions via Problem Context

Another way students utilized the context in reasoning about Task 5 was how they interpreted the negative table values. For some students, like Jake, they could shrug off a potential contradiction. Jake interpreted the table values as representing the number of infected individuals. This means that a negative table value could have served as a contradiction, leading to a theory shift. However, Jake waves away the contradiction by claiming that the negative table values must imply, “there’s like a negative amount of people infected I guess. Um, let’s see, I don’t know just, dropped below the line of infected individuals, I guess maybe they were infected and they died? And they’re still… or maybe they’re people immune.” Jake does not have to settle on any one idea here to disregard the apparent issue. It appears he assumes he must not fully understand the problem and so this allows him to continue with his core assumption that the table values represent the number of infected individuals instead of having to generate a new local theory. This may be related to the difficulties other researchers have identified with how students reason about area under the curve when a function is not strictly positive (e.g., Orton, 1983).

Anne similarly reasons her way through a potential contradiction but instead of attributing the discrepancy to a lack of understanding, she adjusts the problem context entirely to fit within her current theory therefore preserving the viability of her assumption that the table values represent the number of infected individuals. Anne acknowledges that negative people is not a viable interpretation, “so I mean obviously, you don’t have like negative people but, like it’s saying on day zero there was eighteen people…” but instead of altering her theory to increase viability, she alters the problem itself. She claims that the negative five in the table must represent five people who should have been included in the original figures but were not, “I guess if like if they had these people as the original eighteen then they found five people who were sick who weren’t sick anymore they would be like oh, that was five people hadn’t included in the original number that were sick and, but now they’re not sick.” This is a rather sophisticated approach to maintain the consistency of her theory and I believe this creative alteration of the given data is only possible for her because of her confidence with and understanding of the context.

Implications

There is ample evidence in the current study that problem context influenced how the students reasoned about the tasks. For educators, this means that we need to give students ample opportunities to solve accumulation tasks within various contexts. The results of this study indicate that students may only reason about the accumulation in specific ways when given a specific representation. Additionally, we need to be cautious about what kinds of tasks we use in summative assessments in calculus. Assuming a student’s performance on a contextually-based test question accurately models that student’s ability to solve similar tasks in different contexts may not be warranted. Their experiences in their chosen major and their educational history has given them specific tools they will utilize to reason about these tasks.

Viewing students’ work on calculus tasks through the lens of viability is a meaningful way to approach the data analysis. The local theory diagrams were immensely helpful in my attempt to better understand how the students were solving the tasks by developing and revising local theories concerning the tasks. Creating these diagrams provided me the opportunity to view the data through a different lens and thus I came to understand more about how students interpret calculus tasks and what it takes for them to notice a mathematical contradiction. I believe there is merit in the continued development and use of the local theory diagrams in qualitative data analysis both in calculus and more broadly in mathematics education research.

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