Generating Equations for Proportional Relationships Using Magnitude and Substance Conceptions

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We discuss a magnitude conception and a substance conception of fractions and variables that future middle-grades and secondary teachers used when developing and explaining equations for proportional relationships by reasoning about quantities. We conjecture that both conceptions are important for developing equations. The substance conception is useful when a fraction or variable functions as a multiplicand, but not when it functions as a multiplier. The magnitude conception is useful when a fraction or variable functions as a multiplier, but may not be essential when it functions as a multiplicand. Expertise may involve recognizing that the conceptions are distinct and developing a sense of when each conception is useful.

Keywords: Equations, proportional relationships, variables, fractions

The domain of ratio and proportional relationships is a gateway to algebra, other topics in K-12 and undergraduate mathematics, and science (National Center on Education and the Economy, 2013). Yet this crucial domain is also one of the most challenging to learn (e.g., Lamon, 2007). Our research group has been studying how future middle grades and secondary teachers reason about ratios and proportional relationships as they take our mathematics content courses, which focus on multiplicative ideas. In this paper, we are interested in reasoning that takes a *variable-parts* perspective on proportional relationships (Beckmann & Izsák, 2015), a perspective that had been largely overlooked in the research literature, but provides a pathway to developing equations and solving proportions. In these reasoning situations, we are interested in what ideas are useful and generative, and what ideas are especially hard. We discuss a conjecture about two conceptions of fractions and variables—a magnitude conception and a substance conception. Based on preliminary analysis of data, we conjecture that both conceptions play an important role in generating and explaining equations for quantities in a proportional relationship, and that knowing when to use which conception is an aspect of expertise.

Background and Theoretical Perspectives

We view ratios and proportional relationships as part of the multiplicative conceptual field (Vergnaud, 1988)—a web of interrelated ideas that also includes multiplication, division, fractions, and linear relationships. According to Beckmann and Izsák (2015), a quantitative definition of multiplication can organize and connect multiplication, division, and proportional and inversely proportional relationships. We therefore use quantitative definitions of multiplication and fractions as central organizing ideas in our mathematics content courses for future middle grades and secondary teachers.

Quantities and Magnitudes

Measurement includes describing the size of entities (objects or stuff) as some number of a chosen measurement unit, which can be a standard unit, such as a liter, or a non-standard unit, such as a strip drawn on a piece of paper. Although quantities are often described as numbers with units (e.g., CCSS; Common Core State Standards Initiative, 2010), we agree with Thompson (1994) that one need not have selected a specific measurement unit to conceive of an entity as a quantity. In this paper, we define "quantity" to mean an entity that either serves as a measurement unit or could be expressed as some number of another measurement unit, where "some number" means any positive whole, rational, or irrational real number. For example, if a student views one strip drawn on a piece of paper as 2/5 of another drawn strip, then we consider the student to be treating both strips as quantities.

The language of linear algebra may be helpful for thinking about quantities. For each measureable attribute, such as length, weight, or volume, we can associate with that attribute a one-dimensional vector space over the real numbers. Given such a vector space, there is no automatic choice for a basis, and we can work with the vector space without having chosen a basis. Therefore, when we view an entity as a quantity, we essentially consider it as an element of one of these one-dimensional vector spaces, but we need not think of the quantity in terms of a basis for the vector space. When we choose a measurement unit for a given attribute, this measurement unit forms a basis for the one-dimensional vector space, and a quantity can be expressed as a scalar multiple of the basis vector, i.e., the quantity can be expressed as so and so many of the chosen measurement unit. We call this scalar (real number) the magnitude of the quantity with respect to the chosen measurement unit (see also Thompson, Carlson, Byerly, & Hatfield, 2014).

A Quantitative Definition of Multiplication

Although people can use intuitive models to recognize some multiplication situations (e.g., Fischbein, Deri, Nello, & Marino, 1985), if we want students and teachers to be able to make principled arguments for why multiplication applies in a situation, then we need a definition of multiplication. If multiplication is to be understood as a single coherent operation that applies across many different types of situations and across whole numbers, fractions, and decimals, then we need a definition of multiplication that applies to all these cases. One version of a definition we use in our courses for future teachers is as follows. In a situation involving quantities, we say that $M \bullet N = P$ if M is the number of groups in the product amount, N is the number of base units in 1 group, and P is the number of base units in M groups for a suitable base unit, group, and product amount in the situation. We call M the multiplier, N the multiplicand, and P the product; M, N, and P can be non-negative whole numbers, fractions, or decimals. This definition is similar to the one given by Beckmann and Izsák (2015). In some of our courses we have reversed the order of multiplier and multiplicand and written the multiplicand first and the multiplier second. Within a course, we use a consistent order to facilitate clear communication.

This definition of multiplication connects multiplication with measurement (e.g., Davydov, 1992). In the definition, N, M, and P are magnitudes of the quantities "the group" and "the product amount" with respect to the measurement units "the base unit" and "the group." In particular, the multiplier and the product are the results of measuring the product amount in two ways. In some versions of our definition, we clarify the measurement language by defining the multiplicand as the number of base units it takes to make 1 group exactly, the multiplier as the number of groups it takes to make the product amount exactly, and the product as the number of base units it takes to make the product amount exactly.

Reasoning with the definition of multiplication requires organizing and structuring quantities by unitizing, iterating, and partitioning—ideas that have been identified as foundational to multiplicative reasoning in the literature (e.g., Hackenberg & Tillema, 2009). It requires unitizing because N base units form 1 group, so those N base units function as a unit; it requires iterating because if M is 5, one must consider 5 copies or iterates of that group; it requires partitioning because if M is 1/5, one must consider 1/5 of that group, so one must partition the group into 5 equal-sized parts.

A Quantitative Definition of Fraction and Fraction Subconstructs

In our courses for future teachers, we use essentially the same definition of fraction as in the *Common Core State Standards for Mathematics* (CCSS, 2010). We define a unit fraction $1/B$ to be the amount formed by 1 part when a unit amount (or whole) is partitioned into B equal-sized parts. A fraction A/B is defined to be the amount in A parts, each of size $1/B$ of the unit amount (or whole). Therefore, this definition relies on partitioning to form unit fractions and on iterating unit fractions to form both proper and improper fractions. Viewing fractions as obtained by iterating unit fractions can be valuable for students (e.g., Behr, Lesh, Post, & Silver, 1983), and we have found that our future middle grades and secondary teachers reason effectively with this definition.

Various fraction subconstructs or interpretations have been identified in the literature, including the measurement and operator subconstructs (e.g., Behr, Lesh, Post, & Silver, 1983; Kieren, 1976). With the measurement interpretation, fractions can be viewed as plotted on number lines via measurement. To plot the fraction A/B we measure A parts, each of size 1/B of the unit (the interval from 0 to 1). With the operator interpretation, the fraction A/B is seen as a transformation that takes one quantity to another, for example by stretching or shrinking.

Later in this paper we identify substance and magnitude *conceptions* of fractions, which are different from the fraction *subconstructs* in the literature. The magnitude and substance conceptions are essentially orthogonal to the measurement subconstruct, whereas the magnitude conception may be a prerequisite for some instances of the operator subconstruct.

Equations for Proportional Relationships

Proportional relationships in which two unknown quantities are in a fixed ratio can be modeled by equations in two variables, including equations of the form $y = m \bullet x$ or $y = x \bullet m$, where m is a constant of proportionality. By "variable" we mean a letter or symbol that stands for any number from some set (which might not be explicitly specified). Multiplication is numerically commutative, but the multiplier and multiplicand play different roles in quantitative situations. Depending on how the quantities in a situation are structured and organized, one of $y = m \bullet x$ or $y = x \bullet m$ might be better for modeling the situation.

In this paper, we are interested in cases where the constant m is a fraction a/b (so a and b are positive integers). Thus, our quantitative definitions for multiplication and fractions are potentially useful for explaining and generating equations for quantities in a proportional relationship. We are interested in ideas needed to generate and explain equations that relate quantities, especially when the quantities are viewed from the variable-parts perspective (Beckmann & Izsák, 2015), as in the paint task in Figure 1. The 2 parts of blue paint and the 5 parts of yellow paint in that task are all the same size as each other, but that size is unspecified and could vary. The equations $Y = 5/2 \bullet X$ and $X = 2/5 \bullet Y$ (among many others) model the situation in the paint task and fit with the definition of multiplication by taking 1 base unit to be 1 gallon and 1 group to be either all the blue paint or all the yellow paint.

Paint Task: The figure below shows volumes of blue paint and yellow paint. The diagram does not show specific numbers of gallons of paint, but each small part of both strips represents the same number of gallons.

Let's say there are X gallons of blue paint (in all) and Y gallons of yellow paint (in all). How are X and Y related?

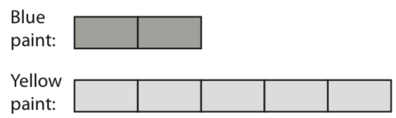

Figure 1. A proportional relationship task about paint, from a variable-parts perspective.

Generating algebraic equations is known to be difficult in part because understanding how algebraic notation symbolizes quantitative situations is difficult (see Kieran, 2007). Even advanced students produce equations with a "reversal error," such as $6S = P$ for a situation in which there are 6 students for every professor (e.g., Clement, 1982). Hackenberg and Lee (2015) explained students' difficulties with generating equations in terms of students' multiplicative concepts, which involve capacities to coordinate multiple levels of nested units and to anticipate, hold in mind, and reorganize such structures. Other authors have pointed to students' conceptions of variables as a source of difficulty, such as treating a variable as a shorthand label for an object or unit (e.g., Küchemann, 1981; Lucariello, Tine, & Ganley, 2014, McNeil et al., 2010). These authors described such a conception of variables as low level or as a misconception. According to Küchemann, using a letter as an object amounts to reducing the letter's meaning from something abstract to something more concrete. He noted that such a reduction often occurs when it is not appropriate, especially in cases where one must distinguish between objects themselves and the number of objects. Yet Beckmann & Kulow (2018) found that future middle grades teachers often used variables as labels when they generated valid equations and produced viable arguments using fractions and multiplication.

A Knowledge-in-Pieces Stance Toward Cognition

We take Knowledge-in-Pieces as our theoretical frame for studying cognition (e.g., diSessa, 1993). In particular, we assume students' knowledge in a mathematical domain is an ecology consisting of many elements, some of which are primitive and intuitive, and simply taken as given, and some of which are more scientific in nature. Some knowledge elements may be closely coordinated, whereas others may be seen as unrelated. Knowledge elements are highly sensitive to context. A knowledge element might be cued in one context but not in another where an expert might view it as relevant. We view learning as a process that involves refinement and coordination of knowledge elements, not a process of repealing and replacing ideas (e.g., Smith, diSessa, & Roschelle, 1993). In particular, this refinement and coordination consists of separating ideas as well as connecting them, and it consists of discerning features of new contexts that make an idea applicable or not applicable, or that make using one idea preferable over another idea (Wagner, 2006). Thus, becoming proficient in generating and explaining equations could involve distinguishing different ways of thinking about a variable or a fraction and a sense of when each way of thinking is more useful or less useful.

Methods, Data Sources, and Research Question

As part of a larger ongoing investigation into future middle grades and secondary teachers' reasoning in the multiplicative conceptual field, we are interested in generating and testing conjectures about ways of thinking about fractions and variables that may be important when developing, explaining, or interpreting equations and expressions involving multiplication. This paper is primarily theoretical because it discusses conjectures we have generated based on initial passes through our data. Our research question for this paper is therefore: Based on our project's data, what ways of thinking about fractions and variables, beyond those already identified in the literature, can we conjecture to be important for generating and explaining equations to relate two unknown quantities that are in a proportional relationship, viewed from a variable-parts perspective?

Data come from 104 semi-structured 75-minute interviews conducted individually with 22 participants, 10 from 2 cohorts of future middle grades mathematics teachers (5 interviews each) and 12 from 2 cohorts of future secondary mathematics teachers (6 with 5 interviews each and 6 with 4 interviews each). All participants were taking mathematics content courses focusing on ideas in the multiplicative conceptual field between the fall of 2014 and the spring of 2017. Interview questions were related to course topics, although some interview questions preceded instruction in a relevant topic. The participants were selected to be mathematically diverse based on their performance on a fractions survey (Bradshaw, Izsák, Templin, & Jacobson, 2014). The data included transcribed video-recording of each interview and scanned copies of the written work each participant generated. To analyze the data, members of the research team watched interviews multiple times, attending to words, gestures, and inscriptions, and wrote cognitive memos discussing and summarizing participants' reasoning.

Conjectures about Conceptions of Fractions and Variables

Based on our initial analysis, we identify two conceptions about fractions and variables—a substance conception and a magnitude conception—that we conjecture play important roles in developing and explaining equations for proportional relationships. To illustrate these conceptions, we use examples that are glosses of reasoning we found across multiple participants, interviews, and interview tasks.

A Substance Conception of Fractions and Variables

A person uses a *substance conception* of a fraction or variable if the person explicitly views the fraction or variable as a label, name, or descriptor of an entity, or as the entity itself. In the case of variables, the substance conception is essentially the same as the label or object conception of variables that has been described in the literature (e.g., McNeil et al., 2010). For example, if a student describes the second strip in Figure 1 as Y and means it as a label or name for the strip, then at that moment, the student is using a label conception of the variable Y. We do not use the term "label conception" because in the case of numbers, we do not want the conception to be confused with cases where a number serves as a non-quantitative label or name, such as a house number or telephone number.

In the case of fractions, if a student describes one of the 5 parts in the second strip in Figure 1 as "a one-fifth-part," or says that the part "is one-fifth," and means that 1/5 is a descriptor or name for the part, or stands for the part itself, then at that moment, the student is using a substance conception of fraction.

We note that the substance conception can also apply to phrases. For example, a student might describe the 5-part yellow paint strip and 2-part blue paint strip in Figure 1 as "the yellow paint" and "the blue paint" respectively, write the equation "the blue paint = 2/5 of the yellow paint," and then write the equation $X = 2/5Y$. In this case, the student uses a substance conception of the phrases "the blue paint" and "the yellow paint," and they might continue to use this substance conception with the variables X and Y. In any case, the student treats the blue paint and the yellow paint as quantities, but they might not be thinking of those quantities as some number of a specified measurement unit, and therefore might not be thinking of X and Y as magnitudes. In essence, the student's equations would be like saying that one vector is equal to a scalar multiple of another vector. In fact, if we interpret X and Y as elements of a vector space, then the equation $X = 2/5Y$ makes perfect sense even if no basis has been chosen for the vector space. So even though we expect the equation $X = 2/5Y$ to be about numbers and to fit with the definition of multiplication, this might not fit readily with a student's interpretation.

A Substance Conception of a Multiplicand may Be Productive. In the example just presented, which led to the equation $X = 2/5Y$, the variable Y functions as a *multiplicand*: it represents 1 group, and 2/5 of that group is the amount of blue paint, X. We conjecture that more generally, when a fraction or variable functions as a multiplicand, a substance conception of a fraction, variable, or related phrase may help the student (1) view the situation in terms of quantities and (2) formulate a correct equation by reasoning about quantities in the situation.

This conjecture is consistent with productive reasoning we have seen with improper fractions. In fact, our definition of fraction almost invites a substance conception. For example, the fraction 5/2 is defined as the amount formed by 5 parts, each of size ½ of the unit amount. According to this definition, 5/2 is essentially the product 5•1/2, where ½ is the multiplicand. Working with the strips in Figure 1, a student might view ½ as a label for each of the 2 parts in the first strip, and also for each of the 5 parts in the second strip. The student might then describe the second strip as 5 parts, each ½, and therefore as 5/2. Even though the student views ½ as a label, the ½ also functions as a quantity for the student because the student considers 5 of the halves. This seems to be a productive way to make sense of improper fractions. What could still be missing, however, is the idea that ½ and 5/2 are magnitudes—the numerical outcome of measurement by the 2-part strip.

A Substance Conception of a Multiplier may Be Unproductive. In contrast, when a fraction or variable functions as a *multiplier*, we conjecture that a substance conception can lead to unproductive interpretations of multiplication. For example, if a student is asked to make a drawing to help explain the meaning of 1/6•X according to our definition of multiplication, the student might draw a 6-part strip, call each part a 1/6-group, and write X in each part, explaining that each 1/6-group has X in it. The student sees each part as 1 group, and sees 1/6 as describing the type of the part, thereby taking a substance conception of 1/6. The substance conception doesn't help the student view 1/6 as *how many* groups are being considered.

This conjecture is consistent with Küchemann's (1981) finding that students were especially challenged to formulate correct algebraic expressions in situations where variables stood for (whole) numbers of objects. The students may have interpreted the variables as the names or types of the objects rather than as their number.

A Magnitude Conception of Fractions and Variables

A person uses a *magnitude conception* of a fraction or variable if the person explicitly views the fraction or variable as a magnitude, i.e., as the result of measuring one quantity by another quantity (which need not be separate from the first quantity). For example, if a student understands that it takes 2/5 of the second strip in Figure 1 to make the first strip, then at that moment, the student is using a magnitude conception of 2/5. Similarly, if a student views Y as the number of gallons of yellow paint in the situation of Figure 1, then at that moment, the student is using a magnitude conception of Y.

A Magnitude Conception of a Multiplicand may not Be Necessary. To use the definition of multiplication as intended does require understanding the multiplicand as a magnitude. However, some students might be able to formulate and explain valid multiplication equations by reasoning about quantities while using only a substance conception of the multiplicand. They might even be able to use the equations by substituting numbers for variables even though they don't think of the variables as magnitudes.

A Magnitude Conception of a Multiplier may Be Necessary. In contrast, when a fraction or variable functions as a *multiplier*, we conjecture that a measurement conception is necessary for a productive interpretation of multiplication. We also conjecture that a measurement conception can be cued by asking a measurement question such as "How many of the second strip in Figure 1 does it take to make the first strip exactly?" A student who answers this question as 2/5 may then see that it takes 2/5 of Y to make X and may therefore formulate the equation $2/5 \bullet Y = X$ even if they have a substance conception of Y and X at the moment.

The Two Conceptions and Moment-by-Moment Reasoning

Finally, we conjecture that the substance and magnitude conceptions are not mutually exclusive. In particular, we conjecture that (1) students can hold the two conceptions simultaneously or that they may switch between the two from one moment to the next, (2) students may not recognize that are using two distinct conceptions when they are reasoning about fractions or variables, and (3) developing expertise with equations involves developing a sense of the difference in the two conceptions and knowing when to use which one.

Conclusion and Future Directions

The future teachers in our mathematics content courses on multiplicative reasoning come to us with various ideas about developing equations, including intuitive or rote approaches, such as setting up an equation of the form $a/b = c/d$ from "a is to b as c is to d." We teach our students to refine their ideas and develop mathematically sound explanations for equations and solution methods by reasoning about how to structure, organize, and relate quantities. To structure, organize, and relate quantities, students must engage with ideas about unitizing, iterating, and partitioning. In addition to these ideas, we conjecture that students also need to refine how they think about quantities, the measurement of quantities, and the mathematical notation we use to describe quantities and their size.

The conjectures we have formulated for this paper come from an initial analysis of a large amount of data. The next step is to find a principled way to select a circumscribed portion of the data for closer examination, so that the conjectures can be put to a rigorous test. We are especially interested in discussions with the audience about this next phase of analysis.

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