Early Undergraduates' Emerging Conceptions of Proof and Conviction

Alison Lynch and Ryan Pugh
California State University, Monterey Bay

Before enrolling in an introduction-to-proof course, undergraduates often hold conceptions of mathematical proof that do not align with those accepted by the mathematics community. These conceptions are informed, in part, by past experiences with proof in mathematics and science courses. In this study, we sought to investigate the influence of these past experiences on students' conceptions of proof. We conducted interviews with nine undergraduates in their first or second year in which we asked them to solve number theory tasks and determine the validity of provided number theory statements. In this paper, we report on the various conceptions of proof these students conveyed and the influence of past experiences on these conceptions.

Keywords: Proof, Conceptions, Student Thinking

Introduction and Motivation

It has been well-established in the literature that undergraduate students struggle to learn to prove. One challenge that students face is that many of them enter university mathematics courses with conceptions of proof that differ from those accepted in the mathematics community. These conceptions include what constitutes a mathematical proof, what purposes a mathematical proof can serve, and how one constructs a mathematical proof. Students develop these conceptions through their past experiences in mathematics, as well as through experience with the idea of proof in non-mathematical settings. Notably, most students in the United States encounter proofs in high school when studying geometry. They also may encounter proofs in a Calculus course, constructing proofs (e.g. epsilon-delta proofs) or making sense of instructor-provided proofs (e.g. the Mean Value Theorem). These experiences influence the way that students conceive of proof in mathematics.

In order to help students develop more robust conceptions of proof, we need to understand the conceptions they bring in with them. In this paper, we explore these emerging conceptions, the factors that influence these conceptions, and the strategies students already employ when determining the truth of a mathematical statement. We ask the following questions:

- How do early undergraduate students' past experiences in math and science influence their conceptions of proof?
- How are early undergraduate students' conceptions of proof related to their strategies for gaining conviction?

Relevant Literature and Theoretical Perspective

Following Thompson's (1992) definition of conceptions of mathematics, we use conceptions of proof to refer to one's "conscious or subconscious beliefs, concepts, meaning, rules, mental images, and preferences" concerning mathematical proof. Conceptions of proof have been studied across populations including high school students (Chazan, 1993; Healy & Hoyles, 2000), undergraduate mathematics majors (Harel & Sowder, 1998; Weber, 2010), and mathematics teachers (Knuth, 2002). In recent years, researchers have also investigated the conceptions of proof held by early undergraduate students - students who have enrolled in at least one college-level mathematics course, but have not yet enrolled in an introduction to proofs course or other proof-based mathematics course (Janelle, 2014; Raman, 2001; Stylianou, Blanton, & Rotou, 2015; Stylianou, Chae, & Blanton, 2006). In the largest of these studies, Stylianou, Blanton, and Rotou (2015) conducted a survey of over 500 early undergraduates about their conceptions of proof, including questions about beliefs and past experiences and multiple-choice proof evaluation tasks. They found that most of the students surveyed selected deductive arguments as the *most rigorous*, but that the proofs students selected as *most explanatory* were the arguments they identified as closest to their own approach (split between deductive, empirical, and narrative). They also found that only a quarter of the students reported having past classroom experiences that "emphasized the importance of developing proofs" (p. 112) and more than half of the students reported past instructors using examples to prove mathematical statements. In this paper, we investigate further the influence of these past experiences on students' beliefs about proof.

When identifying students' strategies for gaining conviction, we focus on *ascertaining*, which Harel and Sowder (1998) define as "the process an individual employs to remove her or his own doubts about the truth of an observation" (p. 241). Much of the existing literature focuses on conviction in terms of what participants identify as convincing in the arguments of others (e.g. Janelle, 2014; Knuth, 2002; Healy & Hoyles, 2000; Chazan, 1993) as opposed to how students construct arguments to convince themselves. Each of these studies found that the majority of participants accepted both deductive and empirical arguments as convincing. However, Stylianou et al. (2015) found that the proofs that students identify as the most convincing and the most like their own approach don't always match their actual proof construction. They gave the same four mathematical statements to 60 students first as proof construction tasks, then as proof evaluation tasks two weeks later. They found that the majority of students constructed empirical arguments, but then reported two weeks later that a narrative or deductive argument was most like what they would construct. Considering this finding, we look at conviction in this paper in the context of students' generated arguments.

Methods

In this study, we conducted hour-long interviews with nine undergraduate students. The participants were all freshman or sophomore students at a small, Hispanic-serving university in the Western United States. The participants were selected because they were enrolled in a college-level mathematics course but had not yet taken an introduction to proof course. Of the nine participants, four were enrolled in Calculus I, three were enrolled in Calculus II, and two were enrolled in Discrete Math. Four of the participants were biology majors, three were marine science majors, and two were computer science majors.

Each student participated in an individual, hour-long, semi-structured interview. During the interview, they were presented with five number theory tasks to explore one at a time. Participants were asked to think aloud as they worked; their speech and writing were recorded using LiveScribe pens, and each interview was videotaped.

On each task, participants were asked if they were convinced by the work they had done and if they considered their work to be a mathematical proof. Depending on their answers, the interviewer asked relevant follow-up questions (e.g. What is missing that would make this a proof? What would you need to do or see to be fully convinced?). After the first task, participants were asked what they believe it means for something to be a mathematical proof. At the end of each interview, participants were asked about their experiences with mathematical proofs in the context of their mathematical careers. Specifically, they were asked if their professors ever show proofs of theorems in class, if they have ever written proofs in their classes or for homework, and what they thought the purpose of proofs in mathematics is.

Tasks

Each participant was asked to work on five number theory tasks, including the three tasks in Table 1. We chose number theory as the content area because it is one of the first topics that students typically encounter in an introduction to proof class. The five tasks were chosen to be easily accessible to the students, requiring only knowledge of divisibility, factors, and even/odd numbers. Some tasks asked students to determine whether a statement was true or false, while others were more exploratory in nature, asking students to create a conjecture.

Table 1. Three of the five number theory tasks used in the study

Task Number	Task Statement
Task 1	Consider the statement: The sum of any 5 consecutive whole numbers is divisible by 5. Is this statement true or false? Would this statement still be true if 5 was replaced with any other number?
Task 3	A factor of a number is a whole number that divides it evenly. For example, the factors of 10 are 1, 2, 5, and 10. Which numbers have an odd number of factors?
Task 4	If a, b, and c are whole numbers, is a times b plus a times c always even, always odd, or can it be either? If b and c are required to be odd, will a times b plus a times c always be even, always be odd, or can it be even or odd?

Data Analysis

For analysis, each interview was transcribed and images of student work from the Livescribe PDFs were added to each transcript. The transcripts were analyzed using a grounded theory approach (Strauss & Corbin, 1994). The two researchers independently coded each transcript using open coding and then discussed themes arising from the generated codes. Codes capturing aspects of participants' conceptions of proof were refined and a modifier was added to capture whether the code was something the participant was convinced by, not convinced by, considered to be necessary/sufficient for proof, or considered not to be necessary/sufficient for proof. Codes were also developed to capture the references participants made to past experiences when discussing proof and conviction. Once the coding scheme was refined, the second author recoded each of the transcripts and wrote a descriptive narrative for each participant. These narratives outlined each participant's ideas surrounding proof and conviction, providing examples and direct quotes from the transcripts to illustrate what they found convincing and what they believed a proof to be.

Results

We report on two aspects of students' emerging conceptions of mathematical proof: the sources they draw upon when forming and articulating these conceptions and the implications for their view of the definitiveness of proof. We also highlight one participant, Rosa, to illustrate the relationship we observed between students' conceptions and the strategies they use to gain conviction.

Influence of Math and Science Experiences on Conceptions of Proof

While the students in our study discussed the concept of proof in mathematics in diverse ways, they drew upon common themes and past experiences as sources for their understanding and reasoning. The three most common themes participants referred to were *Science, High School Geometry,* and *Discrete Mathematics.*

Among the seven science majors (biology, marine science), five drew connections to the study of science in their discussions of mathematical proof. Rosa, José, and Alicia (all Biology majors) described a mathematical proof as only needing evidence – examples or explanations. José described a proof as *"Evidence. Any kind of evidence. Material, biological, any kind of evidence is proof."* Alicia's description of proof was similar: she shared that a mathematical proof is *"show[ing] evidence that it works."* She also talked about more examples being valuable because they served as replication. On Task 4, she gave two confirming examples of the statement, and when asked why two examples were necessary, she explained it as, *"I guess, like, the ability to reproduce the results. Because [the second example] kind of justifies the prior one."* Unlike Rosa and José, Alicia and the two Marine Science majors, Gabriela and Cecilia, described mathematical proof in contrast with ideas from science. Gabriela drew a distinction between the definitiveness of proof in mathematics and in science:

> *Interviewer*: So, in mathematics, what is a proof? What is necessary for something to be a proof?
>
> *Gabriela*: I would say that it's like an absolute thing. And it's been tested many, many times to make sure that there aren't any aren't any exceptions to that one rule. Kind of like how- like it's like a law in science would be.

Alicia drew a similar contrast when comparing proving in mathematics and biology, saying that *"Math is like- I don't know, I feel like once you have it on paper it's pretty much irrefutable, but bio and pretty much it's just, at one point it can be proven wrong."*

Another common theme among the science majors were references to proofs in high school geometry. Four of the seven science majors referred to the two-column proofs they learned in high school, but their interpretations of these proofs differed considerably. Cecilia interpreted the steps in a two-column proof as steps in a deductive argument, describing the second column in terms of logical arguments:

> Well, it's really to prove your logic to get from A to B. It's to show and explain in ways that another person who understands math can look at it, see your work, read that explanation whether it's just, you know, explaining the logic in that one step or actually citing some theorem.

In contrast, Rosa used two-column proofs as justification for why examples and informal arguments were sufficient for proof:

> *Rosa*: Proofs, I get them. I think of geometry. We would get the proofs on one side and, like, have to show that it's true. So proof is making a statement and showing through examples or other like rules that this is a true statement.
>
> *Interviewer*: Oh okay, so you're thinking back to your high school geometry when you were proving properties of triangles and circles.
>
> *Rosa*: Yeah, you'd have like all the true statements on one side and then your work and your explanations on the other side.

The two computer science majors in our study, Antonio and Ana, were enrolled in a discrete mathematics course at the time, and both drew primarily from the content of the course when discussing proof. This is unsurprising since the course contains a week-long unit on mathematical proof, but what was interesting were the features of proof that were most salient to these students. Both students described a proof as a deductive argument which shows that a mathematical statement is always true, but they also emphasized the need for formal language, symbols, and certain structure. For instance, on Task 1, Antonio was fairly convinced that the statement was true from patterns he observed in his examples, but to be totally convinced, he would need to write it the “fancy way”:

Antonio: So, if I did it in the fancy way, with Discrete Math, that's a way to prove how you got the answer.

Interviewer: Yeah?

Antonio: Pretty much. Uh, cause sometimes proving, you need to require some, like, kind of Discrete Math symbols, I say.

The requirement of formal language was restrictive for Ana on Task 3. She had generated a conjecture and articulated an argument in support of her conjecture, but she felt like she lacked the language necessary to write a proof:

Interviewer: Do you think, at this point, based on what you know about this problem, that you would be able to write a proof?

Ana: Probably not. [Laughs]

Interviewer: Why not?

Ana: Because like, I'm just assuming this. I don't know how I would formally write a formal proof. Well, like, when I think proof, it has to be formal, so like, there would have to be, like, if this, then this. And then suppose this. And then you show your proof.

These quotes are representative of a phenomenon we saw broadly in our study of participants using key past experiences as reference points when describing and conceptualizing proof. These references also played a role in how students thought about obtaining conviction, as we discuss more below.

Conceptions of Proof and Strategies for Conviction

Although many of the students in our study described conceptions of proof that differ from the accepted norm in mathematics, there was general consistency between students’ conceptions of proof and how they sought to convince themselves.

The case of Rosa. To illustrate this notion of consistency, we highlight the work of Rosa, a freshman Biology major. As referenced above, Rosa accepted examples and informal explanations as a proof. She described proofs as making a claim and supporting that claim with some evidence. For Rosa, proofs are not definitive:

> Proof is kind of like, I have this idea that, like, you know. […] It would be like [on Task 4], "I think that, you know, when we do the same *a* term with two different *b* and *c* terms, I think we'll always get even" and it could've been true. Like if they were to say I think it's always even and we did these 2 examples right here, we'd be like "Oh, okay." But then I would prove them wrong by saying "Well this one's odd."

For more definitive arguments, she assigned the terms *theory* or *law*. Generalizing from the scientific definitions of the words, Rosa defined theories as, *“things they’ve experimented and it’s been true for the most part. Like, not always, but almost all the time,”* and defined laws as, *“something like gravity, yeah, there’s no proving that it’s not true.”*

Rosa’s beliefs about proof, theory, and law were consistent with what she viewed as convincing. Since proof was not definitive for Rosa, a proof was not necessarily convincing. On Task 1, she tried two examples that both worked and she described her two examples as a proof of the statement. However, she wasn’t convinced that it would always be true:

Interviewer: Okay, are you convinced that it's going to be true for any 5 consecutive numbers?

Rosa: I don't think, um...There's not a lot of absolutes in math, like you know? So I don't know. I'm not convinced that this will always be true, but like for right here it was true.

Rosa was more convinced when she could articulate why a statement was true. On Task 3, she formed a conjecture that the numbers with an odd number of factors were the perfect squares, and she checked her conjecture with three confirming examples (16, 4, and 64). She then explained the rationale behind her conjecture, that the numbers with an odd number of factors are *"the ones where you don't have to list the extra factor"* because the *factors "partner up"*. Rosa was quick to describe this as a proof (since *she "made a statement and then showed examples to why [she] made that statement"*), but later upgraded it to a theory, bordering on a law:

> I would go to say that this one is, like, I did theory on this one because it would be different if I said like you know, I gave a couple of examples but I'm saying like specifically the squares, like the perfect square ones, so I'm already going more in depth and, um, if it was the, yeah theory. See, it's almost on the border of a law.

From her work on these two tasks, we see that Rosa is more convinced by arguments that are more deductive and explanatory in nature, classifying these arguments as theory or law. Since Rosa's *law* is closest to what mathematicians would call *proof*, her reasoning is significantly more mathematically sound than her definition of proof would suggest.

Definitiveness of Proof. We observed another area of consistency between students' conceptions of proof and whether they viewed proofs to be definitive. Of the nine students in our study, four described mathematical proofs as non-definitive (i.e. a proof does not guarantee that a mathematical statement is always true). However, as we saw with Rosa, all but one of the students' views of the definitiveness of proof were consistent with their conceptions of proof. All four of the students who viewed a proof as non-definitive also accepted examples and informal explanations as a proof, whereas four of the five remaining students required a formal deductive argument as a proof.

Implications

In this study, we observed students drawing upon common past experiences in math and science when thinking about and describing mathematical proof. However, despite these experiences being common in a broad sense, different students had internalized different meanings from the experiences. For instance, two students used two-column proofs from high school geometry to support the claim that proofs are deductive, but two other students used the same proofs to claim that examples were sufficient. As a result, it seems that identifying the sources students draw upon in their conceptions is too coarse of a unit of analysis for making sense of those conceptions.

We also observed that, while some students classified empirical arguments as proofs, this does not necessarily mean they viewed those arguments as definitive. In fact, the students' notions regarding what made an argument convincing were far more mathematically accurate than their notions of what constitutes a proof. In future studies, researchers should take care not to conflate these two separate sets of conceptions.

References

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics, 24*(4), 359-387.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education III* (pp. 234-283). Washington, DC: Mathematical Association of America.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education, 31*(4), 396-428.

Janelle, J. (2014). Proof conceptions of college calculus students. In T. Fukawa-Connolly, G. Karakok, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education,* Denver, Colorado (pp. 708-713).

Knuth, E. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education, 33*(5), 379-405.

Raman, M. (2001). Beliefs about proof in collegiate calculus. In R. Speiser, C. A. Maher, & C. N. Walter (Eds.), *Proceedings of the 23rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* Snowbird, Utah (pp. 595-603) .

Strauss, A., & Corbin, J. (1994). Grounded theory methodology: An overview. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 273-285). Thousand Oaks: Sage Publications.

Stylianou, D., Blanton, M., & Rotou, R. (2015). Undergraduate students' understanding of proof: Relationships between proof conceptions, beliefs, and classroom experiences with learning proof. *International Journal of Research in Undergraduate Mathematics Education, 1*(1), 91-134.

Stylianou, D., Chae, N., & Blanton, M. (2006). Students' proof schemes: A closer look at what characterizes students' proof conceptions. In S. Alatorre, J. L. Cortina, M. Saiz, & A. Mendez (Eds.), *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Merida, Mexico (Vol. 2, pp. 54-60).

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). Reston, VA: National Council of Teachers of Mathematics.

Weber, K. (2010). Mathematics majors' perceptions of conviction, validity, and proof. *Mathematical Thinking and Learning, 12*(4), 306-336.