The Generation and Use of Examples in Calculus Classrooms

Vicki Sealey
West Virginia University

Nicole Engelke Infante
West Virginia University

Johnna Bolyard
West Virginia University

Matthew Campbell
West Virginia University

In this paper, we analyze video data of five instructors teaching the Mean Value Theorem in a first-semester calculus course. Throughout the lessons, graphical examples were provided by the instructors and/or the students of functions that satisfied or did not satisfy the conclusion of the Mean Value Theorem. Through the use of thematic analysis, we identified four themes related to emergence and use of examples: who generated the example, who evaluated the example, for what purpose the example was used, and the richness of the example. We emphasize that instruction that leverages student generated examples can provide a great deal of richness in a mathematics lesson and create opportunities to engage students in authentic mathematical activity. This work contributes to an evolving notion of what is entailed in students' active learning of mathematics and the role of the instructor.

Keywords: Example space, Calculus, Mean Value Theorem, Active Learning, Graphical Representations

Although educational research has shown that students develop deeper understanding of mathematics in classrooms where they are actively engaged, lecture is still the primary (and often only) mode of instruction in many collegiate level mathematics courses (Freeman et al., 2014). In this project, we studied five instructors of first semester calculus who were committed to increasing the amount of active learning in their classes. We analyze data from instruction covering the Mean Value Theorem, which provided many graphical examples. While many themes emerged from this data, in this paper we describe instructors' use of graphical examples in covering the Mean Value Theorem. Specifically, we seek to answer the research questions: In what ways are examples generated and used in instruction? What role do these examples play in contributing to an active learning environment?

Literature Review

While many students view examples provided by teachers and texts as templates for solving homework exercises (Lithner, 2003, 2004), examples can play an important role in developing understanding of concepts. Watson and Mason (2005) introduced the notion of learner generated examples (LGEs) and advocated for their power as a tool for deeper learning. Mason and Watson (2008) elaborated:

> ...when a teacher offers an example and works it through, it is the teacher's example. Learners mostly assent to what is asserted. … When learners construct their own examples, they take a completely different stance towards the concept. They 'assert'; they actively seek to make sense of underlying relationships, properties, and structure which form the substance of the theorem or concept. (p. 200)

Mason and Watson (2008) noted "Learners who are encouraged to be creative and to exercise choice respond by becoming more committed to understanding rather than merely automating behavioural practices" (p. 192). To promote creativity in LGEs, students should be encouraged to consider variation. That is, students need to be comfortable asking and exploring questions such as: “What can vary in this problem?” and “To what extent can this aspect of the problem vary?” Watson and Shipman (2008) note that “if students generate examples, reflection on those examples could, through perceiving the effects of the variations they have made, lead to awareness of underlying mathematical structure. ‘Structure’ here means how elements and properties of mathematical expressions are related to each other.” (p. 98) They further indicated that directed example generation, rather than “directionless exploration,” can be a good way to begin understanding concepts.

Through LGEs, a personal example space (PES) is constructed and developed. A PES is defined to be the set of available examples and methods of example construction a learner has at their disposal for solving problems. Sinclair, Watson, Zazkis, and Mason (2011) examined how personal example spaces are structured, paying attention to the varying degrees of “connectedness” such PESs may have. The more connected one’s example space, the greater the likelihood of having a stronger understanding of the concept. They indicate that slightly different prompts may trigger the use of different examples.

Theoretical Perspective

We frame our work considering active learning and the role of examples in the undergraduate mathematics classroom from a communities of practice perspective (Wenger, 1998). The mathematics classroom, as a community, should be a microcosm of the broader mathematics community—engaging in similar disciplinary practices such as proof and justification, seeing structure in mathematics, and the collaborative pursuit of mathematical discovery. What makes the mathematics classroom, whether in K-12 or at the undergraduate level, different from the academic discipline of mathematics is that most of the participants (students) are often newcomers to the taken-as-shared practices, norms, and habits of mind of doing mathematics. However, the classroom community does not (or at least should not) exist in a vacuum—the goal should not only be for students to become more central participants in the classroom (for the sake of the classroom itself) but also in the broader discipline of mathematics, specifically the ways of thinking and reasoning about and communicating with mathematics. Viewing the mathematics classroom as a community of practice, as defined by Wenger (1998), has implications for considering what learning consists of and the role the instructor plays in supporting learning. For our purposes, this perspective also helps clarify some of the structural elements and characteristics of supporting “active learning” in the undergraduate mathematics classroom.

As a social theory of learning, learning from the communities of practice perspective integrates four components: meaning, practice, community, and identity (Wenger, 1998). A productive mathematics classroom is one in which students have the opportunity to learn mathematics. From this communities of practice perspective, this means that students have opportunities to: experience meaningful ways of doing and constructing mathematics (meaning), to then engage in those authentic practices (practice), to be positioned in the classroom community as competent participants in mathematical activity (community), and to come to see themselves (and be seen by others) as one who does mathematics (identity). This multi-faceted process by which newcomers learn and become included in a community of practice is referred to by Lave and Wenger (1991) as “legitimate peripheral participation.” This raises questions about conceptions of active learning that only focus on “participation”—such as opportunities for working in small groups or monitoring air time in whole group contexts. The substance of that participation and how students are ultimately positioned in the midst of that participation is equally important. A focus only on participation may support students coming into the community of the classroom from a purely social standpoint, but be divorced from engaging meaningfully in mathematical ways of working and from being positioned as someone whose ideas are worthwhile, worth building on, and contributing to a collective effort. In our work, we have come to focus on students' opportunities to reason about, offer, and make connections among mathematical examples, and how students have a clear sense of the way in which examples serve a collective effort to build mathematical ideas.

Methods

Subjects in this study were five instructors of first-semester calculus at a large public research university. One of the authors served as the coordinator of the course as well as one of the instructors in the data set. To help preserve confidentiality, we use the term *instructor* to describe the instructor of record of the course, regardless of whether the instructor was a tenured faculty member, a full-time teaching instructor, or a graduate student. We use the pronouns *she, her, and hers* to describe all five instructors, referred to in this paper as Instructor A, Instructor B, etc. All subjects consented to the study, and all but the author received a $500 stipend for their participation at the completion of each semester of the project. Additionally, students in each class signed a media release form granting permission to use their image or voices in our data.

During the first semester of the project, we videotaped class sessions of all five instructors, starting in week three of classes. All sessions that covered new material were recorded, but we did not record sessions when students were reviewing for an exam or taking a quiz or an exam. In each classroom, a video camera was placed in the back corner and was focused on the instructor during the class period. During the second semester, the same five instructors were video recorded when teaching two units, one on the Mean Value Theorem (MVT) which was not coordinated and one on definite integrals, which was highly coordinated. In this paper, we discuss data from the MVT during the second semester. We purposefully selected data from the uncoordinated sessions because this provides an authentic example of instruction in these classrooms without the influence of the coordinated lessons. Three of the instructors covered the material in one day of class, and two of the instructors used two partial days of class.

As part of a larger project, we used thematic analysis, which is a "method for identifying, analysing, and reporting patterns (themes) within data" (Braun & Clark, 2006, p. 79). We employed both theoretical and inductive thematic analysis. Theoretical thematic analysis is "driven by the researcher's theoretical or analytic interest in the area, and is thus more explicitly analyst-driven" (Braun & Clark, 2006, p. 84). Initially, our focus of the analysis was on ways in which active learning was being used in the classroom. As such, we were using theoretical thematic analysis to code for times when students were working in groups or were actively participating in doing mathematics. Moreover, the communities of practice perspective requires that we look not only at the ways in which students are participating, but in the ways that they were meaningfully engaging in mathematics. Thus, we employed theoretical thematic analysis to identify these instances. Additionally, we employed inductive thematic analysis (similar to Strauss and Corbin's (1998) grounded theory) to identify additional themes that were not driven by our own interests. Using both of these techniques, we found instructors' use of examples to be of particular significance. From this, we focused on instances of an example emerging across the five instructors' MVT lessons. Multiple passes through these instances yielded several themes regarding the generation and use of examples—both in isolation and in the context of the full instructional episode.

Data

Recall that the Mean Value Theorem (MVT) says "If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a\ b)$, then there exists a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$" (Larson & Edwards, 2015). A special case of the MVT where $f(a) = f(b)$ is stated in Rolle's Theorem, resulting in the existence of a c value where $f'(c) = 0$. In this section, we first provide a general overview of the five lessons. We then summarize the four themes centered around examples that emerged from our data. Finally, we provide a detailed description of two of the classrooms to illustrate these themes.

During the instruction on the MVT, only Instructor A required students to work in groups during the development of the MVT. Instructors B, C, and D asked their students to work in groups to solve problems related to the MVT after lecturing on the topic. The nature of the worksheets was practicing problems similar to what had been done by the instructor and did not introduce any new material. During all of the lectures, there were many times where the instructors asked students to participate in some way, usually by answering a simple question or verifying that they understood something that was said. Instructors B, C, and E chose to introduce Rolle's Theorem prior to the MVT, while Instructors A and D presented the MVT first, with Rolle's Theorem given as a special case of the MVT. In every lesson, at least five graphical examples were utilized.

Who Generated an Example

The first theme evident in our data relates to *who* generated an example. In all of the classrooms, there were times during a lecture when the instructor would provide an example for the class and write it on the board. We will refer to these instances as *Instructor Generated Examples.* At other times, the instructor asked students to provide an example. In these instances, typically, one or two students provided a response, which tended to be a short one or two-word response. The instructor would then interpret the response and sketch a graph on the board. As such, we call these *Instructor Interpreted Examples.* For example, during Instructor B's lecture, the instructor asked the students if it was possible to draw a graph that was continuous but did not have a horizontal tangent. A student responded with "points," and the instructor drew a graph on the board that resembled $f(x) = |x|$. Often times, *Instructor Interpreted Examples* were in response to questions asked by the instructor that had a very small response space. By this, we mean that the set of possible correct answers is relatively small, and thus, one correct answer often suffices to move the lecture forward. Moreover, it often seemed that the instructors were expecting a specific response to these types of questions and, once the response was given, the instruction proceeded.

Finally, we discuss *Student Generated Examples,* which as the name indicates, are examples that are created by the students. These examples were typically given in response to questions that had a broader response space where there existed several possible correct answers. Most often, these examples were generated when students were given a prompt by the instructor, followed by time to work in groups or to work independently at their seats. For example, during Instructor A's lesson, students worked in groups to create several examples of graphs that did or did not satisfy a list of properties. The students then placed these examples on the board. However, we also saw one case of a *Student Generated Example* given during a lecture, when the instructor asked for an example, and a student responded with $y = x$. While this is still a short response, we claim that the instructor did not need to interpret the meaning of this example, and instead was able to sketch on the board the student's desired graph.

Who Evaluated an Example

A second theme centers around whether the instructor or the students were engaged in evaluating the validity of an example. When an example was presented (by a student or an instructor), it seemed to be assumed that if the instructor put the example on the board, then it was a valid example. One can certainly argue that students should always be evaluating the validity of the examples, and that no audible response from the students does not necessarily indicate that students did not do so. There were certainly times when the instructors asked the students, "Does this work?" or "Does this make sense?" However, in our data, we only saw one chunk of video when students audibly discussed whether or not a graph was a valid example. This happened in Instructor A's class. After the students put their own examples on the board, they were given an opportunity to critique each other's examples and argue whether or not they were valid. However, even in this case, the instructor settled the disagreement and explained why the graph was a valid example.

For What Purpose the Example was Used

We saw two main ways that an example was used. One was to demonstrate an idea or a property. These examples tended to be along the lines of "existence proofs" where one example was enough to demonstrate that something was possible. Another way an example was used was to build an idea and/or to have students discover a concept. In these cases, there seemed to be several examples that were generated, and connections were made across examples. Or, sometimes a specific example was used to address a common misconception. For example, students often mistakenly thought that a linear function did not satisfy the conclusion of the Mean Value Theorem, and both Instructors A and B used a linear example to address this misconception.

The Richness of the Examples

Finally, we note the importance of the richness of the examples that were used in a lesson. Here we consider first if there were any errors in the examples. Instructor D had multiple mathematical errors in her lesson. One small error occurred when a graph she provided did not pass the vertical line test, and thus did not satisfy the basic condition that $f(x)$ be a function. This particular error was not commented on in the class. We also evaluate here examples that may be correct, but perhaps are limited in scope. A deeper discussion of the theme of richness will be provided in the next section.

Classroom Vignette: Instructor A

We discuss Instructor A's classroom, as this lesson demonstrates all of the themes discussed previously. At the beginning of the instruction on the MVT, the graph in Figure 1 was provided to the students as an *Instructor Generated Example*. She then asked her students to tell her what a secant line was (several students responded), and she drew the secant line on the graph between the two endpoints. Next, she told her students to work at their seats to see if there was any place on the graph where there was a tangent line with the same slope as the secant line, and if so, to sketch the tangent line at that point on their own paper. For just over two minutes, the instructor walked around the room, looking at the work done by the students and clarifying directions for students who had questions. We noticed that it seemed to be expected that every student would participate. This is in contrast to the lectures of the other instructors, where one or two students would provide an answer, but the rest of the students would not actively contribute.

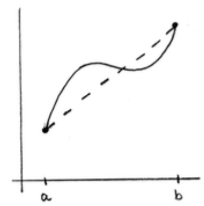

Figure 1: Instructor Generated Initial Example

After a brief class discussion about the previous graph, Instructor A told the class to work in groups to see if they could find examples of other graphs where there was or was not a tangent line with the same slope as the secant line between the two endpoints. She instructed her students by saying, "Your next job is to make sure you find some graphs that do have this property and some graphs that don't have the property." Students spent approximately 16 minutes working in groups to create several *Student Generated Examples* that satisfied the property and several that did not satisfy the property. At one point, the instructor put one of the student's examples on the board (a linear function) and told the class to make sure they discussed an example like this one, if they hadn't already done so. She did not tell them whether or not that graph satisfied the conditions, but expected the students to decide on their own.

After it was clear that every group had several *Student Generated Examples*, she instructed each group to send at least one person to the board to sketch an example of a graph that did not have this property (i.e. a graph where there was no tangent line with the same slope as the secant line between the endpoints). Nine graphs were drawn on the board by the students, and Instructor A added one more graph that was used by one of the groups, but was not the one they chose to put on the board. Thus, there were ten *Student Generated Examples* on the board, a few of which are shown in Figure 2. Next, the class was instructed to look at all of the graphs to see if there were any graphs that should not have been on the board, so in other words, to see if any of the graphs on the board had a place where the slope of the tangent line was equal to the slope of the secant line. This created an opportunity for the students to evaluate the validity of the examples.

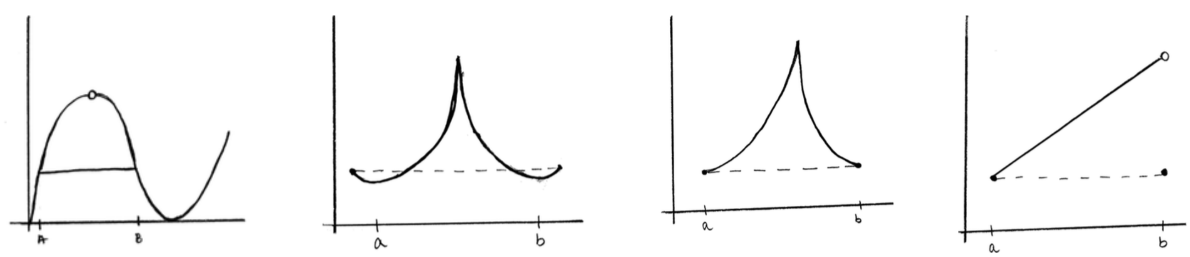

Figure 2a *Figure 2b* *Figure 2c* *Figure 2d*

Figure 2: Student Generated Examples

When discussing the *Student Generated Examples*, three interesting things happened. First, one student argued that the graph shown in Figure 2a was wrong because there is a place outside of the interval with a horizontal tangent line. The instructor clarified that the task was only to attend to whether or not the property held *on* the interval from *a* to *b*. Next, another student questioned the graph in Figure 2a because he recognized that even though the function was not defined at one point, it looked like the limit would still exist. At this point, the instructor led the class in a nice discussion about the definition of the derivative and why tangent lines do not exist at places where there is a removable discontinuity. Third, Instructor A pointed out that the graph shown in Figure 2b was not quite accurate, even though the students' intent was correct. She cautioned the students to be careful with their graphs and make sure that their examples clearly illustrated the intended properties, then modified the graph to form the example in Figure 2c.

Next, the instructor led the class in a discussion about what the ten graphs on the board had in common. First, she highlighted the seven graphs that had some sort of discontinuity, and asked the students what the other three graphs had in common. At least one student responded that those graphs had a point or a cusp, and Instructor A introduced the term *differentiable* and emphasized that all of the graphs that were drawn without a tangent line parallel to the secant line were either not continuous or not differentiable. Then, the instructor gave the class a short period of time to think about graphs that are both continuous and differentiable on the interval to decide if those graphs had to have a place where the tangent and secant lines were parallel.

The purpose of the examples that had been generated was illustrated as the instructor wrote the MVT on the board and related it to what the students had created. For example, when stating that the function must be continuous on the closed interval [a, b], she referred to the example in Figure 2d to illustrate that an open interval would not have guaranteed that the property held. This example was a rich example that nicely illustrated this concept. In contrast, Instructor B had an *Instructor Generated Example* on the board that was extremely similar to Figure 2d, but she did not discuss why this example illustrated the need for the function to be continuous on a closed interval. Furthermore, Instructor D, claimed that a closed interval was required so that it would be possible to compute the average rate of change. As indicated by Figure 2d, Instructor D's statement does not justify the need for continuity on a closed interval.

Discussion and Teaching Implications

In a classroom that supports students' mathematical learning in a way consistent with the communities of practice perspective, the instructor is also tasked with supporting newcomers in engaging with and becoming more skilled with disciplinary practices. This has implications for the way in which the instructor represents mathematics (for example, the role of examples in developing mathematical ideas) and how the instructor engages students meaningfully in that effort as well. We want to emphasize that this does not simply mean that students should have more opportunity to work in groups or that students should talk more during class; instead, we emphasize that the nature of the task must provide students with the opportunity to *deeply* explore mathematical concepts. In our data, a simple prompt from Instructor A afforded her students the opportunity to deeply engage in developing the Mean Value Theorem. Watson and Shipman (2008) sum this up as:

> ...significant learning can result from the process [of generating examples] because learners generate and explore example spaces related to the ideas, in particular spaces of relations between objects. The importance of normal classroom expectations and teacher guidance cannot be overestimated here. (p. 108)

We also emphasize that the task of generating the examples, while extremely important for student learning, is not all that is necessary. The instructor also needs to be skilled in leading a discussion about the examples in a way that moves the lesson forward. He or she needs to know which examples to highlight in order to provide richness as well as to demonstrate concepts. Our focus on the generation and use of examples contributes to a sense of what is entailed in students' active learning in mathematics. These findings have implications for how instructors can be supported—through materials, coordination, or instructional support—to create classroom environments that actively engage students in doing mathematics.

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