Peter's Evoked Concept Images for Absolute Value Inequalities in Calculus Contexts

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Statements involving absolute value inequalities, such as the definition of continuity at a point, abound in Advanced Calculus. In textbooks, such statements are frequently illustrated with graphical representations. Despite their abundance, how students think about absolute value inequalities and their representations in these contexts is not widely known. This study examines one undergraduate mathematics student's evoked concept images (Tall & Vinner, 1981) for absolute value inequalities in various contexts, including those from Advanced Calculus. The student's evoked concept image differed based on the context of the statements involving absolute value inequalities. Notably, the student's evoked concept image did not support his understanding of the visual representation of the formal definition of continuity. The results of this study suggest that some students may not conceive of absolute value inequalities in ways that are productive for understanding the formal definitions of Advanced Calculus concepts.

Keywords: Absolute Value Inequalities, Calculus, Visual Representations

Absolute value inequalities are used in numerous formal definitions and theorems central to advanced Calculus, including statements involving limits, continuity, and sequence convergence. For example, the formal definition of continuity at a point, historically attributed to Weierstrass, may be stated as: "A function f is continuous at a point c in its domain if, for each real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that, for all x in the domain of f with $|x - c| < \delta$, $|f(x) - f(c)| < \varepsilon$." Not much is known about how students conceive of absolute value inequalities in such statements from advanced Calculus. While research has examined students' understanding of absolute value inequalities, most studies have addressed students' procedural fluency and their common errors at lower levels (Almog & Ilany, 2012). Additionally, many high school algebra textbooks that introduce absolute value inequalities treat them procedurally, instructing students to consider cases of inequalities (Boero & Bazzini, 2004). Conceiving of absolute value inequalities primarily in terms of the algebraic procedure for finding a solution may be insufficient for making conclusions from statements involving absolute value inequalities, such as those commonly used in Advanced Calculus. Furthermore, a procedurally-oriented conception may not be sufficient to support students in understanding graphical representations of statements such as the definition of continuity at a point. For example, several Analysis texts introduce the formal definition of continuity of a function at a point along with an image like the one shown in Figure 1 (Gaughan, 1997).

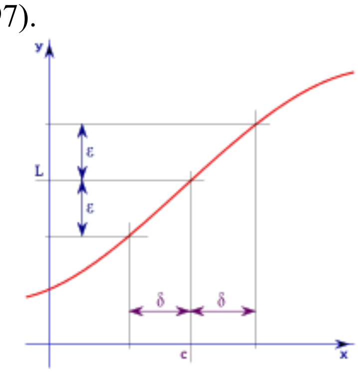

Figure 1. A visual representation of continuity at a point

A student that only has a procedural meaning for absolute value inequalities, such as $|x - c| < \delta$, may not necessarily associate the values of *x* that satisfy this inequality, with the values of *x* within a distance of δ from *c* on the *x*-axis in Figure 1. In graphical representations of advanced Calculus statements, solutions to absolute value inequalities typically refer to a region of space in the rectangular coordinate system with points whose coordinates are within a certain distance from a point. Several studies have found that conceptualizing an absolute value as a distance on a number lines helps students visualize the solutions of an absolute value inequality, thus developing a critical conception of absolute value statements at lower levels (Curtis, 2016; Sierpinska, Bobos, & Pruncut, 2011). The aim of this study is to extend the research in this area by characterizing students' meanings for absolute value inequalities like those found in statements from advanced Calculus, particularly with regard to associated visual representations.

Specifically, the research question for this study is as follows: *What meanings for absolute value inequalities are elicited for students in the context of advanced calculus statements?*

Theoretical Perspective

In this report, I adopt a constructivist perspective, consistent with von Glasersfeld's (1995) view that students' knowledge consists of a set of action schemes that are increasingly viable given their experience. In this view, students construct knowledge for themselves, and words and images do not inherently contain meaning. This viewpoint also implies that I, as a researcher, do not have direct access to students' knowledge and can only model student thinking based upon their observable actions and behaviors. To characterize student meanings for absolute value inequalities in this study, I also adopt Tall and Vinner (1981)'s constructs of *concept image* and *evoked concept image*. By *concept image*, Tall and Vinner (1981) refer to "the entire cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). Thus, a student's concept image for absolute value inequalities may include numerous cognitive processes and images built on various experiences with the topic over time. While a student may have a concept image that contains many properties and processes for absolute value inequalities, in a given context, only parts of this concept image are activated at a given time. Tall and Vinner (1981) thus define *evoked concept image* to refer to the aspects of the concept image accessed within a particular context. They also note that different aspects of a students' concept image may be in conflict with one another, without the student's awareness.

Hypothesized Productive Meanings for Solutions to Absolute Value Inequalities in Advanced Calculus Statements

Based on how absolute value inequalities and their solutions are currently utilized in communicating ideas of advanced Calculus, such as the definition of continuity at a point, one productive way of thinking about absolute value inequalities is in terms of bounded distances. For instance, students may understand that solutions to an absolute value inequality of the form $|x - c| < \delta$ can be determined by finding all values of *x* that are within a distance of δ from *c* on a number line. In two dimensions, this set of solutions is a region of points whose *x* values are within a distance of δ from *c* on the *x*-axis. Coming to such an understanding involves connections between a relationship represented algebraically and a set of solutions represented geometrically. Acquiring this level of understanding can be complex, requiring understandings of many foundational ideas, such as variable and difference.

The solution to $|x–c| < \delta$ can be represented analytically as $c–\delta< x < c+\delta$ or geometrically as all x values within a distance of δ from c. Connecting this inequality successfully to the graphical representation involves students viewing both the algebraic inequality and graph as representing an upper bound on how much x can differ from c. That is, they must see that the solutions can be represented by an interval on a number line that includes all values (represented by the letter x) within δ of a value represented by c. They must conceptualize the letter c as representing a central value, and the δ symbol as representing an upper limit on the solutions' distance away from c. For example, the solution set to an inequality like $|x –(–1)| < 3$ can be represented as follows:

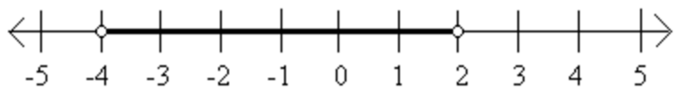

Figure 2. One-dimension representation of solutions to $|x –(–1)| < 3$

In this representation, values within a distance of 3 from –1 are included in the set of solutions to the inequality.

In the Cartesian plane, rather than an interval on a number line, this solution set is represented by a region marked by vertical lines representing the boundaries of this solution set. Thus, the region would include all points whose x-coordinate is at most a distance of 3 away from –1. Similarly, with inequalities of the form $|f(x)–f(c)| < \varepsilon$, the two-dimensional representation of $f(x)$ values (represented on the vertical axis) that satisfy this inequality can be represented by a horizontal region. This solution set is represented by a region marked by horizontal lines representing the boundaries of this solution set. Thus, the region would include all points whose y-coordinate is at most a length of 1 away from 3.

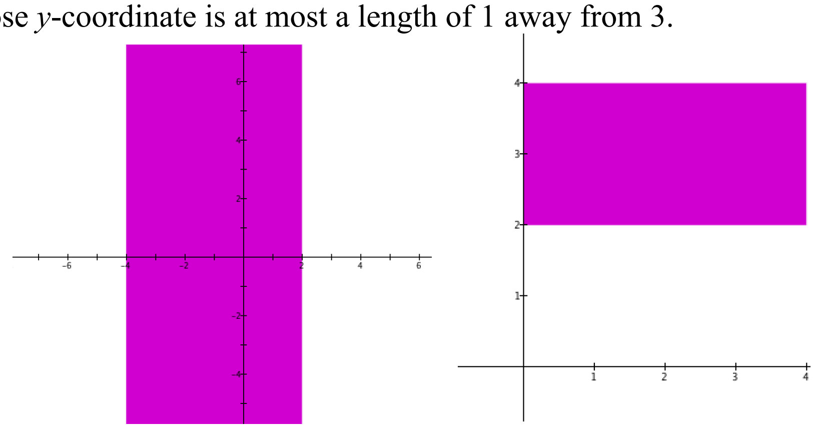

Figure 3. Two-dimensional representation of solutions to $|x +1| < 3$ and $|f(x)–3| < 1$

Methods

For this study, I conducted one 120-minute clinical interview (Clement, 2000) with an undergraduate mathematics student, Peter. Peter was a math major who had completed the Calculus sequence and an Introduction to Proof course, but had not yet taken an Advanced Calculus course.

In the interview, Peter was given tasks that were designed to elicit his meanings for absolute value, absolute value equations, absolute value inequalities, including associated visual representations, such as representing solutions on a number line. Because of the hypothesized evoked concept images for each task, the tasks were ordered in such a way that earlier tasks would not be influenced by later ones. One of the earlier tasks involved a statement about a function f that was the formal definition for continuity at the point $x = 1$ as shown below:

For each real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that, for all x in the domain of f with $|x - 1| < \delta$, $|f(x) - f(1)| < \varepsilon$.

After asking the student to explain the statement in his own words, as well as each portion of the statement, I presented him with two graphs, first Figure 4 (left) and then Figure 4 (right). After presenting each graph, the student was asked to evaluate whether the statement was true or false for the function f shown in each graph and explain his reasoning.

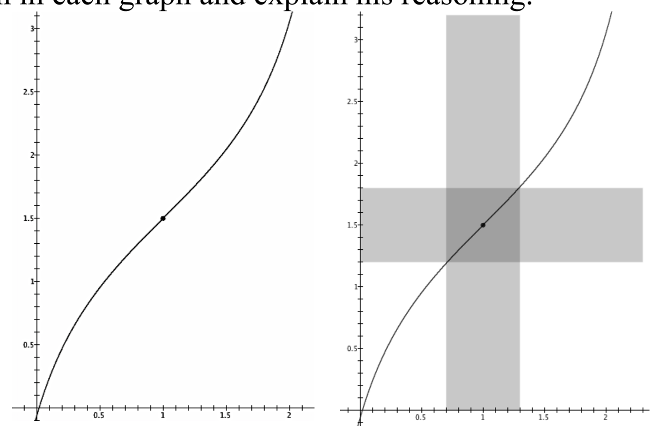

Figure 4. Graphs used with statement of continuity at x=1

The final task given to the student is shown in Figure 5.

a. Find a pair of values, a, b such that $|a - b| = 3$.
b. Find another pair of values a, b with $a < 0$ such that $|a - b| = 3$.
c. Find another pair of values a, b with $a < b$ such that $|a - b| = 3$.
d. If $a = 1$, how many possible values of b satisfy the statement?
e. What must be true about these pairs of values?
f. Can you use a number line to explain what $|a - b|$ represents?
g. Can you use a number line to explain what $|a|$ represents?

Figure 5. Final task presented to student in interview

The purpose of this task (Figure 5) was to examine how the student solves absolute value equations, and how he explains the meaning of solutions to absolute value equations involving multiple variables. After the interview, I analyzed the data by modeling Peter's evoked concept image of absolute value (inequality) in each task, especially looking for distinctions in the types of images evoked between contexts.

Results

In this section, I report several key responses to tasks that revealed Peter's evoked concept image for absolute value and absolute value inequalities. Early in the interview, Peter's written work and utterances suggested that his initial evoked meaning for absolute value was "that a value is positive." When presented with the formal definition of continuity at a point, Peter expressed some confusion, and acknowledged that he was not sure what the statement meant. When presented with the first associated graph (see Figure 4, left), he labeled the graph as shown in Figure 6 below. Peter's procedural meaning for absolute value, that is, making values positive, led to him representing the absolute value of a difference on each axis as a single value.

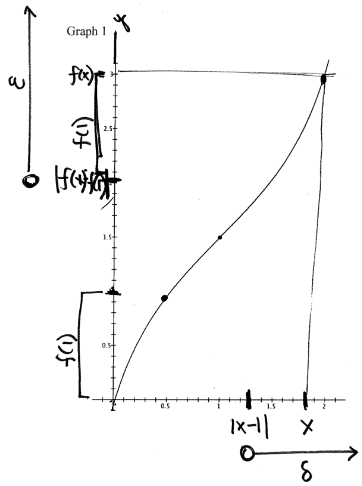

Figure 6. Peter's labels on the graph of a function related to the formal definition of continuity at a point

When the interviewer asked Peter to explain "$|x–1|<\delta$," on this graph, Peter responded by saying,

> "So let's say I just chose some value of x here (*labels x on the x-axis as shown in Figure 6*), then $x–1$ (*labels |x–1| on the x-axis to the left of x*), (*pauses*) then δ would have to be larger than that (*draws ray with open circle and labels it "δ"*), so uhh all the values…delta could possibly be any value along this interval (*points to ray he just drew*)."

Peter's words and labels suggest that he was conceptualizing $|x–1|$ as a value on the x-axis to the right of zero and one unit to the left of x. When prompted to explain what the inequality "$|x–1|<\delta$" represented graphically, Peter provided a literal interpretation of the symbols in his response, stating that δ had a value greater than the value of $|x–1|$. He illustrated this on the number line by constructing a ray with an open circle at $|x–1|$ extending to the right on the x-axis. Notably, when shown the graph in Figure 4 (right) and asked to explain the statement relative to the image, Peter paused for a long period of time and acknowledged that he was not quite sure how the statement related to the graph of the function and the shaded regions.

Later in the interview, working on Task 8, Peter's work indicated a different evoked concept image for absolute value that included a distance from zero. In this task, Peter was asked to compare the values of "$|a+b|$" and "$|a|+|b|$" using a number line. Peter produced the following illustration:

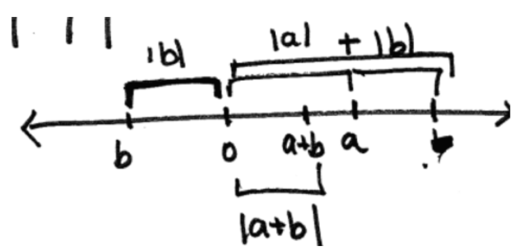

Figure 7. Peter's work on Task 8

In Peter's work (Figure 7), he label *a, b,*and *a+b* at different locations on the number line, and then labeled line segments from 0 to respective places on the number line with absolute values. Peter placed $|a+b|$ on the segment starting at 0 and ending at $a+b$. The labeling suggests that he considered $|a+b|$ as the distance $a+b$ was from 0 on the number line. Additionally, Peter's work on this task shows his attention to distances. Peter independently chose b to be to the left of 0, and a to be to the right of 0, farther away than b was from 0. When considering $a+b$, Peter attended to the placement of this value relative to the distance a and b were from 0. That is, since a is farther to the right of 0 than b is to the left of 0, $a+b$ would have a positive value less than the value of a, and Peter placed $a+b$ to the left of a but to the right of 0. Peter's work on this task was the first indication that he was using absolute value to represent a distance from 0 on a number line.

In the final task, Peter's meaning for absolute value of a difference shifted from his previous meaning. Earlier (as shown in Figure 6), Peter labeled the absolute value of a difference at a location on the number line, indicating he was thinking of a single value that was the result of taking the absolute value of a difference. In the final task, in answering part e) "What must be true about these pairs of values [that satisfy $|a–b|=3$]," Peter's evoked concept image shifted.

To answer this question, Peter first wrote out two equations, "$a–b = 3$" and "$a–b = –3$" and solved them in terms of b and then in terms of a. Peter then explained "If I were to choose a, then b would be either 3 away from a or 3 on the other side of a" and drew a number line to illustrate his idea, as shown below in Figure 8.

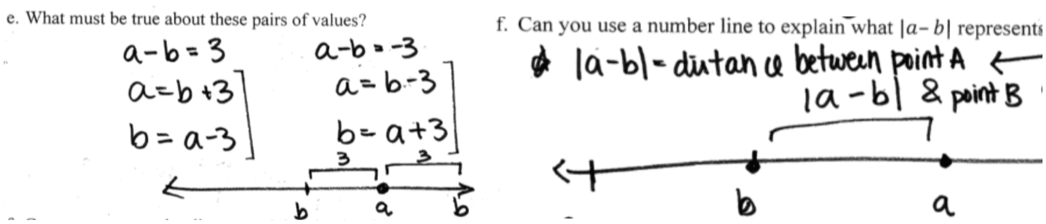

Figure 8. Peter's work showing what must be true about a and b when |a–b|=3

Peter illustrated his algebraic interpretation of the relationship between a and b using a number line. He labeled the segment he drew between a and b in either direction with "3," indicating that he recognized the distance between a and b was 3 units. This was the first time that Peter interpreted a difference as a distance between two points, neither of which were 0. When Peter encountered absolute values of differences when responding to earlier tasks, he considered them to be single values on the x-axis, or measuring a distance from 0. Rather than treat an absolute value of a difference as a single value whose reference point is 0, Peter treated an absolute value of a difference as a distance between the two values, without reference to 0. In the next sub-question in the final task, Peter confirmed that he was conceptualizing the absolute value of a difference $|a–b|$ as the distance between point a and point b (Figure 8, right).

To check to see if his image for absolute value inequality that had been evoked in the task above influenced his understanding of continuity at a point, I again showed Peter the formal definition of continuity at a point, and associated graph, and asked him to re-label the graph.

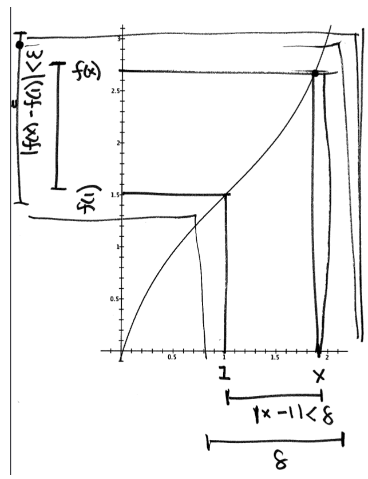

Figure 9. Peter's updated labels on graph after completing final task

This time, Peter labeled the distance between x and 1 on the x-axis with the label $|x–1|$ rather than labeling $|x–1|$ as a value on the x-axis itself. He recognized that the statement "$|x–1|<\delta$" was a statement about a comparison of a distance between two values, and a value δ. While Peter did not attend to x values to the left of 1, his evoked image for absolute value of a difference as measuring a distance (Figure 8) supported Peter in connecting the image of a graph with the definition of continuity at a point.

Conclusion & Discussion

Peter's work suggests his concept image for absolute value and absolute value of a difference contains several distinct meanings and processes. In different contexts throughout the interview, Peter's work indicated different evoked concept images for absolute value, consistent with findings by Tall and Vinner (1981). In the beginning of the interview, Peter's meaning for absolute value inequalities elicited by the initial tasks included a procedure of making a value positive. Later in the interview, Peter's evoked concept image for absolute value included a meaning of distance from zero on a number line. In the final task, Peter's evoked concept image for absolute value of a difference was a distance between two points.

Most notably, Peter's initial meaning for absolute value elicited by the continuity at a point statement and associated graphs did not include a difference between two points on axes, but rather were of absolute value as an operator that makes values positive. Peter's initial evoked image is consistent with the way absolute value inequalities are introduced in high school textbooks (Boero & Bazzini, 2004). Due to his evoked meaning for absolute value as an operator, Peter was unable to explain the continuity statement relative to the graphs in Figure 4. However, through other various tasks, different aspects of Peter's concept image for absolute value were evoked, which allowed Peter to conceptualize the absolute values in the continuity statement differently than he had previously. The findings from this study suggest that students entering advanced Calculus courses may interpret absolute value inequalities and their visual representations differently than intended. Specifically, their evoked concept image for absolute value may not support their attempts to connect such statements to associated graphs. Instructors of courses utilizing statements involving absolute value inequalities may consider including tasks to evoke different meanings for absolute value. Instructors and curriculum developers should not assume that students' evoked concept image for absolute value inequalities will align with how their solutions are represented in illustrations on graphs.

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