Teacher Learning About Mathematical Reasoning: An Instructional Model

Robert Sigley
Texas State University

First, I describe an instructional model for Teacher Learning about Mathematical Reasoning (TLMR), designed for pre-service (PSTs) and in-service teachers (ISTs) to: (a) build knowledge of the various forms of mathematical reasoning that students naturally make use of in their justifying solutions to problems, (b) attend to the development of students' mathematical reasoning from studying videos and student written work, and (c) learn about the conditions and teacher moves that facilitate student justifications of problem solutions. Second, I provide a detailed description of activities from a representative cycle of the TLMR model. Finally, I report briefly on preliminary results indicating teacher growth in identifying and recognizing student reasoning for PSTs and ISTs who underwent the TLMR model compared to a comparison group.

Keywords: mathematical reasoning, pre-service teachers, video-based intervention

As the knowledge required for effective mathematics teaching has become more clearly defined, knowledge needed for PSTs and ISTs to attend to student reasoning in justifying solutions to problems has been recognized as essential (e.g., Francisco & Maher, 2011). In fact, earlier work on student reasoning has been influential in shaping US national policy by developing a set of Standards for Practice in identifying behaviors that are desirable for students as they engage in doing mathematics. (NCTM, 2014). It is essential that PSTs and ISTs become knowledgeable of these practices so that they can attend to and encourage student mathematical reasoning in their classrooms. In addition, while models on how they can do this have been theorized, there is little research testing the effectiveness of these models. In this paper, I will describe an intervention model that is designed to promote teachers' attention to student reasoning through: (a) open-ended problem solving of mathematical tasks, (b) studying videos of children building solutions to the same tasks, and (c) in-class and online discussions about their own and students' problem solving, as well as relevant readings and the results of a quantitative study to test the effectiveness of the model in having PSTs and ISTs attend to student mathematical reasoning.

Children's Mathematical Reasoning and Justification

Research on the knowledge for teaching mathematics identifies that knowledge of students' mathematical reasoning is essential (Ball, 2003) and that there is a relationship between following how students builds their knowledge and their performance in mathematics (Rowan et. al, 1997). An important early finding in research into mathematical reasoning is that in a natural way, children – even young children – build proof-like justifications, providing convincing arguments that take the form of reasoning by cases, induction, contradiction, and upper and lower bounds (Maher & Martino, 1996). Students' justifications are driven by an effort to make sense of the problem situation, notice patterns, and pose theories (Mueller, Yankelewitz, & Maher, 2012). Three key components surface in promoting the development of mathematical reasoning. These are: (a) reflecting on and revisiting earlier mathematical concepts, (b) collaborating and discussing strategies and modes of solution; and (c) engagement in open-ended tasks that elicit justification (Maher, Powell, & Uptegrove, 2010). Research results across all ages and contexts, formal and informal, indicate that certain tasks tend to elicit particular forms of reasoning (e.g., upper and lower bound arguments, inductive arguments) when students are encouraged to provide a justification for their solutions (Yankelewitz, Mueller, & Maher, 2010).

Teacher Learning About Mathematical Reasoning: A Model

TLMR was implemented as a design-research study in a graduate mathematics education course over 5 years, with each implementation co-taught by the author. The problems used during class were from the domain of counting and combinatorics. Each iteration was over a 14-week period. Overall, 86 teachers participated. The model has six cycles of interventions with each cycle containing five components. These are: (a) teacher collaborative problem solving (b) teacher study of videos of children working on the same problems, (c) teacher analyses of samples of student written justifications of the same problems, (d) teacher small-group online discussions (with guiding questions) designed for synergistic reflection on their own and students' problem solving, in light of the assigned readings, video study, personal experience, and collaborative problem solving. Throughout the course, teachers engaged in each of the six cycles by first working collaboratively on the related mathematical tasks. I offer an example for each of the five components of a cycle.

A Sample Cycle

The Tasks: The TLMR intervention utilized tasks from earlier research studies that were shown to elicit student reasoning in justifying solutions to problems. All tasks were introduced prior to the mandated school curriculum. During this cycle, teachers worked on two variations of pizza problems: Pizza with Halves, selecting from two toppings; Whole Pizzas, selecting from four toppings. They were asked to discuss the strategies used in solving both problems, attend to similarities or differences, compare their strategies, reflect on strategies for previous problems, and report findings to the entire class.

The Videos: Teachers were then assigned to study two video clips online, read and react to two articles, and examine four samples of student work from fifth graders working on the same Pizza with Halves Problem. The first video, http://dx.doi.org/doi:10.7282/T3HM57PQ, followed twelve, fifth-grade students across two class periods as they worked on the Pizza Problem with Halves selecting from two toppings. The clip showed students constructing various representations to justify their solution of the ten pizzas. The second clip, http://dx.doi.org/doi:10.7282/T3VX0FRD, was a task-based interview with fourth grader Brandon. It shows Brandon, explaining his solution to the Pizza Problem selecting from four toppings (without halves). After explaining his solution, Brandon is asked whether this problem reminds him of any other problem he had worked on earlier and he said it reminded him of the Towers Problem, a problem where he had to find the unique number of towers he could build four high selecting from two colors of Unifix cubes. After resolving the Towers problem, he makes a connection between the similarity in structure of the two problems, recognizing that the two choices for a pizza topping (represented by 1 or 0) for being on or off the pizza is similar to the two choices for the color of a particular block of the tower, e.g. red or yellow (PUP Math, 1999).

The Readings: The first readings discussed details of the Brandon Video (Maher and Martino, 1998). The second reading dealt with the topic of isomorphisms in mathematics education (Greer and Harel, 1998). The Maher and Martino, paper situated the Brandon video as a part of a longer study and included details that preceded the interview as well as an analysis of Brandon's problem solving. The Greer and Harel paper referred to Brandon as an example of a nine-year old student having an insight in recognizing an isomorphism, similar to the mathematician, Poincare.

Student Work: The student work module contained four pieces of student solutions from the Pizza with Halves problem, selecting from two toppings. These were chosen to illustrate the variety of representations and arguments produced by the students. The teachers were asked to review the students' representations and work and specifically address: (1) the correctness of the solution, (2) description of the strategy, (3) the validity of the reasoning, and (4) whether or not they find the solution convincing and, if so, why. If they did not find the solution convincing, they were asked to indicate from studying the student work what pedagogical moves they might take to help the student develop a convincing argument.

Online Discussion: For this module, the guiding questions focused on the notation that Brandon used in his problem solving. Teachers were asked to discuss how, if at all, Brandon's choice of notation was helpful to him in recognizing the relationship between the Pizza and Tower problems. They were also asked to discuss the forms of reasoning displayed by Brandon in the video, and the role of isomorphisms in mathematical cognition. Finally, they were asked to compare their own problem solving with that of the students in the Pizza with Halves and in the Brandon video.

The Study

Limitations in time and space allow only a brief description of preliminary results. A reasoning assessment (RA) was administered as a pre-test before the course and as a post-test after the course. The group that underwent the intervention above was 86 teachers over the five iterations. Additionally, pre and post data was collected from 48 teachers from the same course taught by a different instructor. The comparison group's course had the same emphasis on attending to mathematical reasoning, but did not use the TLMR model. The comparison group is used to check if the students just got more on the post-test because they watched the video twice, if that is true, we would expect to not see a significant difference in the post-test scores between the comparison and experimental group. The RA consisted of a ten-minute video of fourth-graders sharing their arguments for an open-ended problem-solving task. Teachers were asked to: (a) identify the arguments presented by the children, (b) determine the validity of the arguments, (c) provide evidence to support their claims, and (d) explain whether or not the arguments were complete. The clip contained arguments by induction, cases, an alternate cases argument, and contradiction. The responses were scored by a group of three people using an established rubric to determine if the participant noted no features, partial features (the alternate cases and contradiction responses did not contain a partial feature, the other two did), or complete features of the argument. Initially, the group worked with responses from a pilot study for training and to establish reliability. For the analysis, growth was defined as recognizing more features on the post-test than the pre-test. Each argument was analyzed using a Wilcoxon rank-sum test, a nonparametric alternative to the t-test to determine if the growth from pre-to-post was significant. The effect sizes were calculated using an estimator suggested by Grissom and Kim (2012) which takes the U statistic generated by the test and then divides it by the product of the two sample sizes that will estimate that a score randomly draw from one population will be greater than the other. This methodology was chosen over using Cohen's d due to the smaller sample size and non-normality of the data.

Results

The pre-test results for the comparison and experimental group were not significantly different and the differences in each iteration for the experimental group was not significant, so the experimental group was put together into one group, giving a size of 86 teachers. For the first cases argument, on the pre-test the experimental group had 50% missing the argument, 2.3% had a partial argument, and 47.7% had the complete argument compared to 61.2% missing, 6.1% partial, and 32.7% complete for the comparison group. On the post-test, in the experimental group 21.6% was missing the argument, 1.1% partial, and 77.3% complete compared to 57.1% missing, 8.2% partial, and 34.7% complete. The growth from pre-to-post for the experimental group was significant with a moderate effect size ($p < 0.01$, effect size = 0.351) and for comparing the experimental to comparison post with a moderate effect size ($p < 0.01$, effect size = 0.248). For the alternate cases argument, the experimental group went from 69.3% missing on the pre-test to 33% on the posttest and 30.7% complete on the pre-test to 67% complete on the posttest. This growth was significant ($p < 0.01$, effect size = 0.277) and the post score compared to the comparison group was significant ($p < 0.01$, effect size = 0.32). The inductive argument for the experimental group had 55.7% missing, 37.5% partial, and 6.8% complete on the pre-test and 23.9% missing, 53.4% partial, and 22.7% complete on the post-test for a significant growth ($p < 0.01$, effect size = 0.317), and a significant difference in post compared to the comparison group ($p < 0.01$, effect size = 0.324). Finally, the argument by contradiction was missing from 95.5% of the experimental teachers' pre-test and complete in 4.5% compared to missing in 68.2% of the post-test with 31.8% complete for a significant growth ($p < 0.01$, effect size = 0.364) and significant difference than the comparison post results ($p = 0.011$, effect size = 0.402).

Discussion and Implications

The design TLMR research study produced an extensive and valuable database about teachers learning to attend to student reasoning. Preliminary analyses suggest significant changes in teacher beliefs in recognizing the potential for student reasoning. In addition, there is evidence that teacher recognition of the forms of arguments used by children in the video as they expressed their justifications of solutions improved over the course of the intervention (Maher et. al, 2014). Analysis of teacher online discussions of one cycle of intervention also indicated some interesting findings. As teachers compared their own problem solving with that of (a) other teachers, (b) students from the videos, and (c) student work samples, they pointed out similarities and differences, especially when the representations differed from their own and were in their view, "more elegant". These comparisons prompted further reflection about what constitutes a convincing argument in posing a solution to a challenging mathematical task and raised expectations about students' creativity in representing their solutions. It is interesting that teachers focused heavily on attending to details in the videos, relating their observations to their own personal experience. Reference to the readings was also made by teachers who tried to situate their learning within a particular theoretical perspective. Implementation of the TLMR holds promise for teacher growth in attending to the development of mathematical reasoning in students. Further studies building off this work can focus on the application of teacher learning through the TLMR model to their practice.

Questions

(1) I collected some qualitative data as well (online discussions, in-class problem solving) – would those help strengthen the argument that the model is successful in promoting attending to reasoning?

(2) Is it worthwhile to follow PSTs into their practicum and initial classroom experience to see how it affects them or would it be too messy?

References

Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A Research Companion to principles and standards for school mathematics* (pp. 27-44). Reston, VA: National Council of Teachers of Mathematics.

Francisco, J. M. & Maher, C. A. (2011). Teachers attending to students’ mathematical reasoning: Lessons from an after-school research program. *Journal of Mathematics Teacher Education*, 14 (1), 49-66.

Greer, B. and Harel, G. (1998). The role of isomorphisms in mathematical cognition. *Journal of Mathematical Behavior*, 17(1), 5-24.

Grissom, R. J., & Kim, J. J. (2012). *Effect sizes for research: Univariate and multivariate application*. (2nd ed.). New York, NY: Taylor & Francis.

Maher, C. A. & Martino, A. M. (1996). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education*, 27(2), 194-214.

Maher, C. A., Powell, A. B., & Uptegrove, E. (Eds.). (2010). *Combinatorics and reasoning: Representing, justifying and building isomorphisms*. New York: Springer Publishers.

Maher, C. A., Palius, M.F., Maher, J. A., Hmelo-Silver, C. E., & Sigley, R. (2014). Teachers Can Learn to Attend to Students’ Reasoning Using Videos as a Tool. *Issues in Teacher Education*, 23(1), 31-47.

Mueller, M., Yankelewitz, D. & Maher, C. (2012). A framework for analyzing the collaborative construction of arguments and its interplay with agency. *Educational Studies in Mathematics*, 80(3), 369-387.

Rowan, B., Chiang, F. S., & Miller, R. J. (1997). Using research on employees' performance to study the effects of teachers on students' achievement. *Sociology of Education*, 70(4), 256-284.

Yankelewitz, D., Mueller, M. & Maher, C. (2010). A task that elicits reasoning: A dual analysis. *Journal of Mathematical Behavior*, 29(2), 76-85.