Katlyn's Inverse Dilemma: School Mathematics Versus Quantitative Reasoning

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In this report, I examine the interplay between Katlyn's (an undergraduate student's) inverse relation (and function) meanings developed through her continued school experiences and her reasoning about relationships between quantities. I first summarize the literature on students' inverse function meanings and then provide my theoretical perspective, including a description of a quantitative approach to inverse relations (and functions). I then present Katlyn's activities in a teaching experiment designed to support her in reasoning about a relation and its inverse relation as representing an invariant relationship. Although she engaged in such reasoning, her continued school mathematics experiences constrained her in reorganizing her inverse function meanings. I conclude with a discussion and areas for future research.

Keywords: Inverse Function, Inverse Relations, Preservice Teacher Education

Researchers examining students' quantitative reasoning (Thompson, 2011) have found that students can develop foundational meanings for various concepts such as linear (Johnson, 2012) and exponential functions (Ellis, Ozgur, Kulow, Williams, & Amidon, 2012) by reasoning about relationships between quantities *before* developing more formal mathematical understandings. In contrast, examinations of students' inverse function understandings have found students often maintain disconnected inverse function meanings after they have received instruction (Brown & Reynolds, 2007; Kimani & Masingila, 2006; Vidakovic, 1996). I conjectured quantitative reasoning could potentially support undergraduate students relating and connecting their inverse function meanings developed through their school experiences. Working with a pre-service teacher, Katlyn, who had K-14 school experiences with inverse function, I investigated how she could potentially re-construct her inverse function meanings via her reasoning quantitatively. In this report, I present Katlyn's progress in a semester-long teaching experiment intended to investigate the question: How does a student's quantitative reasoning interplay with her inverse function meanings developed through her continued school mathematics experiences?

Research on Inverse Function

Vidakovic (1996) proposed that students develop inverse function schemas in the following order: (a) function, (b) composition of functions, then (c) inverse function through a coordination of (a) and (b). Whether implicitly or explicitly, many researchers (Brown & Reynolds, 2007; Kimani & Masingila, 2006; Vidakovic, 1996) examining students' inverse function meanings have emphasized composition of functions and the formal mathematical definition (i.e. f and f^{-1} are inverse functions if $f(f^{-1}(x)) = f^{-1}(f(x)) = x$) as paramount to students developing productive inverse function meanings. However, these and other researchers have found students often maintain disconnected (from the researcher's perspective) inverse function meanings, often related to executing certain activity in analytic rule or graphing representations (Brown & Reynolds, 2007; Kimani & Masingila, 2006; Paoletti, Stevens, Hobson, LaForest, & Moore, 2015). For instance, students often use a "switching-and-solving" technique when determining the inverse function of a given function represented by an analytic rule (i.e., given $y = x - 2$ they switch the variables and solve for y to obtain $y = x + 2$) but are experience difficulties interpreting the results of this activity for a contextualized analytic rule. The extent to which students relate their switching-and-solving technique or their other activities to function composition is an open question. The current body of research indicates that current approaches to teaching inverse function have been ineffective in supporting students in developing interrelated inverse function meanings. In this report, along with Paoletti (2015), which I elaborate on below, I begin to address the evident need to re-conceptualize ways to support students developing sophisticated inverse function and inverse relation meanings.

Theoretical Framing

I examined the possibility of supporting students developing inverse relation (and function) meanings via their reasoning about relationships between quantities. A *quantity* is a conceptual entity an individual constructs as a measurable attribute of an object or phenomena (Thompson, 1994, 2011). An individual constructs *quantitative relationships* as she associates two varying (or non-varying) quantities (Johnson, 2012; Thompson, 1994). As an individual constructs and analyzes these relationships, she engages in *quantitative reasoning* (Thompson, 1994).

Specific to inverse relations, I conjectured if a student constructed a (non-causal) quantitative relationship between two quantities (e.g., quantities A and B), then she could decide to consider one quantity as the input of a relation (e.g. B input, A output) whilst anticipating the other quantity would be the input of the inverse relation (e.g., A input, B output). By focusing on the underlying quantitative relationship, the 'function-ness' of a relation and its inverse falls to the background; a student can describe and represent a relation and its inverse without (necessarily) being concerned if either represents a function. Further, the student maintains an understanding that choosing input-output quantities does not influence the underlying relationship that the associated relations or functions describe.

I conjectured a student maintaining such understandings can interpret a single analytic rule or graph as simultaneously representing a relation and its inverse relation. With respect to graphing, the student anticipates that either axis can represent the input quantity of a relation. Although this reasoning may seem insignificant, Moore, Silverman, Paoletti, & LaForest (2014) illustrated that students' graphing meanings are often restricted to reasoning about the input quantity exclusively represented on the horizontal axis.

In Paoletti (2015), I demonstrated the feasibility of this way of thinking by presenting one undergraduate student's (Arya's) activities as she reorganized her inverse function meanings compatible with this description. When addressing inverse function tasks in a pre-interview Arya relied on switching techniques (e.g., switching-and-solving) and understood a function and its inverse represented different relationships. Throughout the teaching experiment Arya experienced several prolonged perturbations. Resolving these perturbations supported her in reorganizing her inverse function meanings as well as her meanings for variables and graphs. Specifically, at the conclusion of the study, Arya understood that a relation and its inverse relation, regardless of 'function-ness', represented an invariant relationship. Keeping this invariant relationship in mind, Arya understood a single graph could be interpreted as either a relation or its inverse by choosing either quantity represented on either axis as a relations input. In context, she made sense of the switching-and-solving procedure by changing the quantitative referent of each variable when switching variables (i.e. if V represented volume and s represented side length in the original analytic rule then V represented side length and s represented volume in the inverse analytic rule). Arya's meanings at the conclusion of the study demonstrate both the viability of the ways of thinking described above and one way students can relate their quantitative reasoning to the switching-and-solving procedure.

Methods and Task Design

I conducted a semester-long teaching experiment (Steffe & Thompson, 2000) with Katlyn and Arya (pseudonyms), two undergraduate students enrolled in a secondary mathematics teacher education program. The students were juniors who had successfully completed at least two courses beyond a calculus sequence. I engaged the students in three individual semi-structured clinical interviews (Clement, 2000) and 15 paired teaching episodes. Clinical interviews and teaching episodes provided flexibility to create and adapt tasks to explore how students might develop meanings compatible with those I described. Specifically, I used clinical interviews as one pre and two post interviews to develop models of Katlyn's mathematics (Steffe & Thompson, 2000), including her quantitative reasoning and her inverse function meanings, without intending to create shifts in her meanings. I used teaching episodes to examine the viability of my hypothesized models and to pose tasks I conjectured might create perturbations for Katlyn, possibly leading her to make accommodations to her meanings.

I analyzed the data using open (generative) and axial (convergent) approaches (Strauss & Corbin, 1998) in combination with conceptual analysis (Thompson, 2008). I developed and refined models of Katlyn's mathematics by initially analyzing the videos identifying episodes of Katlyn's activity that provided insights into her meanings. These instances supported my generating tentative models of her mathematics that I tested by searching for corroborating or refuting activity. When Katlyn exhibited novel activity, I adjusted my models to explain this activity including the possibility that this activity indicated fundamental shifts in her meanings. Through this iterative process of creating and adjusting hypotheses of Katlyn's mathematics, I was able both to characterize her thinking at a specific time and to explain shifts in Katlyn's meanings throughout the teaching experiment.

I first raised the notion of inverse function in the *Graphing Sine/Arcsine Task* (Figure 1). The research team designed this task to support students in developing inverse relation meanings compatible with those described above. The first two prompts ask students to create graphs of the sine (Graph 1) and arcsine, or inverse sine, (Graph 2) functions. The third prompt asks the students to consider how they could interpret Graph 1 as representing the arcsine function. This prompt also asks the students to consider if Graphs 1 and 2 represent "the same relationship." I conjectured asking the students to foreground the "relationship" represented by the graphs might support them in conceiving either quantity, on either axis, could represent the input of a relation in order to conceive Graph 1 as representing both the sine and arcsine functions or relations.

Graph 1:	Create a graph of the sine function with a domain of all real numbers. What is the range?
Graph 2:	Using **covariation talk**, create and justify a graph of the arcsine (or inverse sine) function.
Prompt 3:	Can you alter (**do not draw a new graph**) Graph 1 such that it represents the graph of the arcsine function? Does this graph convey the same relationship as the second graph? How so or how not?

Figure 1. The Graphing Sine/Arcsine Task.

Results

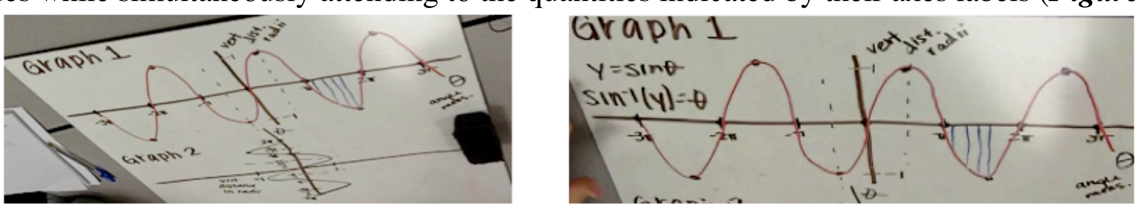
I first present analysis from the initial clinical interview that provides insights into Katlyn's inverse function meanings relevant to this report. I then present her activity addressing the prompts in the Graphing Sine/Arcsine Task. I conclude with Katlyn's activity in the final clinical interview, highlighting the interplay between her quantitative reasoning and her inverse function understandings developed through her continued school mathematics experiences.

Results from the Initial Clinical Interview

During the initial interview Katlyn's predominate meaning for inverse functions involved "switching." For example, given the equation $C(F) = (5/9)(F - 32)$ defining the relationship between degrees Celsius and degrees Fahrenheit, Katlyn switched C and F and solved for C determining the inverse rule $C^{-1}(F) = (9/5)F + 32$. Given a line representing the relationship between temperature measures, Katlyn estimated values of several coordinate points then switched abscissa and ordinate values to determine points on a line she drew to represent the inverse function. In both cases, Katlyn was uncertain how to interpret the results of her activity in relation to temperature measures indicating she did not attend to the underlying quantities when engaging in these techniques. For example, Katlyn identified that the point (10, 50) on the given line represented that 10 degrees Fahrenheit corresponds to 50 degrees Celsius but when interpreting the point (50, 10) on her constructed line Katlyn said, "My whole reasoning in this entire process… is switching x and y, is switching C and F which is how I came up with this graph. So I don't necessarily know what... the new graph would stand for." Katlyn did not assign any meaning in relation to temperature measures to the point (50, 10) on her constructed graph representing the inverse function.

Reasoning about the Sine and Arcsine Relationships

In the first four teaching episodes the students represented the relationship between angle measure and vertical distance above the horizontal diameter in a circular motion context and understood this relationship was defined by the sine function, compatible with the descriptions of Moore (2014). After these episodes, I prompted the students with the Graphing Sine/Arcsine Task. They carefully attended to the quantitative relationship between angle measure and vertical distance as they created Graph 1, then constructed Graph 2 by switching abscissa and ordinate values while simultaneously attending to the quantities indicated by their axes labels (***Figure 2***a).



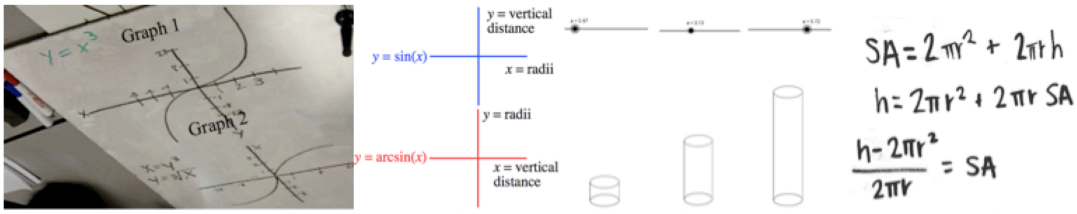
(a) (b)

Figure 2. The pair's (a) Graph 1 and Graph 2 and (b) Graph 1 with added equations.

Having drawn both graphs, the pair set out to address Prompt 3. Katlyn wrote $y = \sin(\theta)$ next to Graph 1, indicating this was the equation they initially represented with Graph 1. She then added $\sin^{-1}(y) = \theta$ below $y = \sin(\theta)$ (see added labels in ***Figure 2***b). Katlyn anticipated considering vertical distance as her input, represented on Graph 1's vertical axis, stating, "We're looking at the y [*pointing to y in* $\sin^{-1}(y) = \theta$], so we go to one [*motioning to 1 on the vertical axis*] and then we're like okay well… which angle's sine is one?" Katlyn motioned horizontally to the three points on Graph 1 with a vertical distance value of one. Continuing to explain her reasoning, Katlyn said, "If we're switching the input and output… So we want theta to be our answer, 'cause then originally theta was our input but now we want it to be our output." As in the initial clinical interview, Katlyn referred to "switching" but in this episode she maintained her focus on the invariant relationship between vertical distance and angle measure as she considered how to interpret Graph 1 as representing a relation with vertical distance as the input quantity. Katlyn reasoned quantitatively to consider a relation and its inverse relation as representing an invariant relationship but with different chosen input and output quantities.

Considering a Decontextualized then Contextualized Relationship

Because the students never referenced switching-and-solving when working with the sine and arcsine relations, two teaching episodes later, I asked the pair to address the prompts in Figure 1 for a decontextualized function ($y = x^3$) to investigate if, and if so how, their activity would be different for a decontextualized function. The students drew Graphs 1 and 2 (see Figure 3a) by maintaining the relationship between x and y (e.g., they described that for $x > 0$ with x represented on the horizontal and vertical axis in Graph 1 and Graph 2, respectively, y increased at an increasing rate with respect to x). However, the students experienced a perturbation as this graph, which they understood was defined by $x = y^{1/3}$, was not defined by the analytic rule, $y = x^{1/3}$, they had determined by switching-and-solving. This perturbation led the students to question their prior activity with the sine and arcsine relations in which they did not switch-and-solve.



(a) (b) (c) (d)

Figure 3. (a) The pairs decontextualized graphs, (b) the color-coded axes with Katlyn's added labels, (c) the cylinder animation, and (d) Katlyn's work.

Intending to maintain the students' focus on quantitative relationships, I contextualized this function as representing the volume and side length of a cube ($V = s^3$) as I conjectured they would not switch the variables to represent the inverse rule in a context. I asked the pair what the inverse rule would be and Katlyn immediately responded "cube root of V equals s." I repeated, "cube root of V equals s," to which Katlyn refuted, "No, but that's not right." Katlyn experienced a perturbation as she oscillated between her switching technique and maintaining the relationship between volume and side length while maintaining the quantitative referent of each variable.

I asked Katlyn if she knew why she switched variables and she responded, "No, I just remember doing that, that's just our definition… you like switched x and y and solved for y again because in standard position y is [on the vertical axis] and x is [on the horizontal axis]." I considered her argument of "standard position" of x and y as a way to support Katlyn in relating her switching technique and maintaining the underlying quantitative relationship. Drawing attention to the possibility of using variables to arbitrarily represent quantities values, I wrote $y = \sin(x)$ in blue and $y = \arcsin(x)$ in red along with Cartesian coordinate axes next to each (Figure 3b). For each graph, I asked Katlyn to identify the variable and quantity each axis would represent if she were going to graph each rule. Responding to this, Katlyn used the variables x and y arbitrarily to define angle measure and vertical distance to represent the input with the variable x on the horizontal axis in each graph (see black labels in Figure 3b).

After this Katlyn described her reasoning about the inverse function in the side length-volume context, arguing, "We've just been saying like we need to switch them in the equation [*pointing to* $y = x^{1/3}$] but like, we're like switching them in real life." Katlyn then reasoned she had to reassign the quantitative referents of the variables when switching-and-solving (i.e. s represented volume and V represented side length in $V = s^{1/3}$). In the moment, Katlyn understood that in a contextualized situation (e.g., sine and arcsine, volume and side length) a relation and its inverse represented the same quantitative relationship but with different input quantities; she reassigned the quantitative referents of each variable when switching in order to maintain this relationship.

Results from the Final Clinical Interview

Based on the described teaching sessions, which spanned two weeks, I conjectured Katlyn potentially reorganized her meanings such that she understood a relation and its inverse represented an invariant relationship with the difference being which quantity she chose to represent the input. I intended to test this conjecture in an interview two months after the last teaching episode addressing inverse relations. I showed Katlyn an applet displaying a cylinder with varying height and a constant radius (Figure 3c) and asked her to determine a relationship between the cylinder's surface area and height. Reasoning quantitatively, Katlyn described imagining the net of the cylinder composed of two circles with constant area and a rectangle with varying area (i.e. h varies and r is constant) and determined the analytic rule $SA = 2\pi r^2 + 2\pi rh$. She drew a linear graph and described the relationship stating, "As like the height is increasing, surface area is also increasing." Conjecturing Katlyn was capable of considering surface area as the input, I asked, "Is there another way to read [the graph]?" Katlyn responded, "As surface area increases, height increases.... whatever happens to one is like happening to the other one." Although Katlyn chose to consider height first, she anticipated this was only one of the options; from my perspective Katlyn reasoned about a relation and its inverse relation as she anticipated coordinating either quantity varying first.

I asked Katlyn to determine the inverse analytic rule conjecturing she would maintain the relationship she had described. However, Katlyn switched-and-solved (Figure 3d). I asked Katlyn to "talk me through what you did there", and she responded:

Katlyn: It's funny that you say that 'cause I'm tutoring two girls and we were doing inverses yesterday. And I don't, and I still can't explain why we do this. I was trying to think of a way to explain it to them, and I didn't know the answer. Um [*pause*]. Because that's what I've been told to do for six years…

TP: Okay. So you said you were just tutoring someone on this?

Katlyn: Yeah, and… they were just like, 'well how do I do it?' And so I told them, like you have to make sure the… function is one-to-one so like for every… input there's only one output and for every output there is only one input. All that nonsense that doesn't, I don't really know why we do that. But that's what has to happen before you can switch your input and output and then solve. So, why do we do this? I don't know. But I know this is what the answer is and I. Yeah, I don't know.

TP: Okay and so this is the answer [*pointing to* $SA = (h - 2\pi r^2)/(2\pi r)$]?

Katlyn: Yes. Yeah, yeah.

TP: But, you're sort of also acting like there's something you're not comfortable with about it.

Katlyn: I just don't know what it means, like I don't, why do I care about this [*pointing to* $SA = (h - 2\pi r^2)/(2\pi r)$]?

TP: So say a little bit more what do you mean you don't know what this [*pointing to* $SA = (h - 2\pi r^2)/(2\pi r)$] means?

Katlyn: I don't know what it means. I know [$SA = (h - 2\pi r^2)/(2\pi r)$] is the inverse, for surface area of a cylinder. That is all I know. Why is it the surface area? What does it, what does the inverse for surface area mean? I guess I'm thinking like. [*pause*] Okay, it reminds me of that time that we were doing like volume of a cube being like side-squared and then we switched the two and then I was like, okay so now, s means volume and V means side[length]. So now does here, [*pause*] surface area mean height and height mean surface area? Or did we just not finish the problem in class to conclude about what, I don't, I don't remember. I have no idea why we do this.

TP: So, you're starting to say here [*pointing to* $SA = (h - 2\pi r^2)/(2\pi r)$]. If, if SA… represented height, and h represented surface area?

Katlyn: Well, it wouldn't make any sense. Because then it would just be the same. Like if you multiplied [$SA = (h - 2\pi r^2)/(2\pi r)$] all back out you would get [$SA = 2\pi r^2 + 2\pi rh$], I guess. And so like I'm attributing [$SA = (h - 2\pi r^2)/(2\pi r)$] to be the same thing where this is now height [*pointing to SA in* $SA = (h - 2\pi r^2)/(2\pi r)$] and this is now surface area [*pointing to h in* $SA = (h - 2\pi r^2)/(2\pi r)$]. That doesn't make any sense. We might as well have kept it that way [*indicating* $SA = 2\pi r^2 + 2\pi rh$]. [*pause*] That's probably not right then cause it has to mean, it has to mean something different.

From my perspective Katlyn described the relation and its inverse prior to the term "inverse" being raised but reverted to switching-and-solving when asked about the "inverse". She engaged in this technique, which she learned as a student and was reinforced as a tutor, despite her being reflectively aware that she did not know *why* she engaged in this activity (e.g., "So, why do we do this? I don't know") or *how to interpret* the activity's results (e.g., "*I just don't know what it means*… why do I care about this"). Katlyn recalled the volume-side length situation from months earlier and considered switching the quantitative referent of each variable. However, she rejected this as the inverse rule would represent the same relationship as the original rule leading her to conclude a function and its inverse function must represent different relationships (e.g., "That doesn't make any sense. We might as well have kept it that way").

Discussion and Concluding Remarks

Katlyn's story exhibits difficulties students may encounter when attempting to reason about relationships between quantities by leveraging their non-quantitative mathematical meanings. Compatible with Arya (Paoletti, 2015), Katlyn reorganized several of her meanings during the teaching experiment (i.e., using variables arbitrarily to represent quantities). However, these reorganized meanings did not lead to shifts in her inverse function meanings. One possible explanation is that Katlyn engaged in in-the-moment activity (potentially both in the study and in her tutoring) to assuage a perturbation without reflecting on if her activity was related to other contexts or situations.

Despite not reorganizing her meanings, Katlyn's inverse function meanings at the end of the study were not significantly different than other students' meanings researchers have characterized (Brown & Reynolds, 2007; Kimani & Masingila, 2006; Paoletti, Stevens, Hobson, LaForest, & Moore, 2015; Vidakovic, 1996). Thompson, Phillip, Thompson & Boyd (1994) distinguished between teachers maintaining calculational and conceptual orientations, noting the latter "focus students' attention away from thoughtless application of procedures and toward a rich conception of situations, ideas and relationships among ideas" (p. 86). If a teacher maintains inverse function meanings similar to Katlyn's, she will be unable to support her students in developing a rich conception of relationships among ideas and instead will have to focus on a thoughtless application of the switching-and-solving technique (i.e. "I was trying to think of a way to explain it to them, and I didn't know the answer…"). Hence, future researchers should continue to address calls (Thompson, Phillip, Thompson & Boyd, 1994; Thompson, 2008) for increased focus on ways of reasoning that support future teachers development of rich conceptions of ideas and relationships among ideas that they can call on in their teaching.

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References

Brown, C., & Reynolds, B. (2007). *Delineating Four Conceptions of Function: A Case of Composition and Inverse*. Paper presented at the 29th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education, Lake Tahoe, NV.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 547-589). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Ellis, A. B., Ozgur, Z., Kulow, T., Williams, C., & Amidon, J. (2012). Quantifying exponential growth: The case of the jactus. In R. Mayes & L. L. Hatfield (Eds.), *Quantitative Reasoning and Mathematical Modeling: A Driver for STEM Integrated Education and Teaching in Context* (pp. 93-112). Laramie: University of Wyoming.

Johnson, H. L. (2012). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *The Journal of Mathematical Behavior, 31*(3), 313–330.

Kimani, P. M., & Masingila, J. O. (2006). Calculus Students' Perceptions of the Relationship among the Concepts of Function Transformation, Function Composition, and Function Inverse. *Conference Papers -- Psychology of Mathematics & Education of North America*.

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education, 45*(1), 102–138.

Moore, K., Silverman, J., Paoletti, T., & LaForest, K. (2014). Breaking Conventions to Support Quantitative Reasoning. *Mathematics Teacher Educator, 2*(2), 141-162.

Paoletti, T., Stevens, I. E., Hobson, N. L. F., LaForest, K., & Moore, K. (2015). *Pre-service teachers' inverse function meanings*. Paper presented at the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education, Pittsburgh, PA.

Paoletti, T. (2015). *Reasoning quantitatively to develop inverse function meanings*. Paper presented at the Thirty-Seventh Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, East Lansing, MI.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–307). Hillside, NJ: Erlbaum.

Strauss, A. L., & Corbin, J. M. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Thousand Oaks: Sage Publications.

Thompson, A. G., Philipp, R. A., Thompson, P. W., & Boyd, B. A. (1994). Calculational and conceptual orientations in teaching mathematics. In A. Coxford (Ed.), *1994 Yearbook of the NCTM* (pp. 79-92). Reston, VA: NCTM.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179-234). Albany, NYY: SUNY Press.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sépulveda (Eds.), *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31-49). Morélia, Mexico: PME.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e* (pp. 33–57). Laramie, WY: University of Wyoming.

Vidakovic, D. (1996). Learning the concept of inverse function. *Journal of Computers in Mathematics and Science Teaching, 15*(3), 295–318.