Student's Semantic Understanding of Surjective Functions

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Reasoning and proof are essential to mathematics, and surjective functions play important roles in every mathematical domain. In this study, students in a transition to proof course completed tasks involving composition and surjective functions. This paper explores students' semantic understandings of surjective functions, both individually and in the context of composition of functions. Most students demonstrated productive semantic understandings of surjective functions that allowed them to produce counterexamples and arguments for the truth of statements. Furthermore, in the struggle of using the syntactic definition of surjective in a proof, some students used their semantic understanding to try to make sense of the definition. This demonstrates the potential of students' ability to reason semantically to build understanding of the syntactic definition and structure of proofs of surjective functions.

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Reasoning and proof are fundamental aspects of mathematics on which mathematical teaching and learning should focus. Weber and Alcock (2004, 2009), describe two distinct reasoning styles and approaches to proof production that they call *semantic* and *syntactic*. Semantic reasoners produce proofs through a focus on general understanding guided by examples, diagrams, or other informal explanations, and syntactic reasoners produce proofs mainly through deductive reasoning based on axioms, definitions, theorems, and standard proof frameworks (Weber & Alcock, 2004). Although a mathematical proof is a syntactic product, understanding the proving process involves both types of reasoning. Thus, "neither of these approaches should be used exclusively by students and both syntactic and referential [semantic] approaches to proving are necessary for proving competence" (Alcock & Weber, 2010, p. 96).

Students typically encounter surjective functions for the first time in precalculus. Although they are not necessarily emphasized at this level, surjective functions are important in upper-division courses as bijections and isomorphisms permeate nearly every mathematical domain. My students consistently struggle with proofs of statements involving surjective functions, so as a step toward understanding why, this paper addresses the following research questions: In what ways do students approach proofs of statements involving surjective functions? What are students' semantic understandings of surjective functions?

Literature Review

Both semantic and syntactic reasoning present students with opportunities and difficulties in proof production. Semantic reasoning can provide a basis for and support the development of a syntactic proof or counterexample by suggesting a main idea or underlying structure (de Villiers, 2010, Moore, 1994; Raman, 2003; Weber & Alcock, 2004). However, students often do not make these connections due to inaccurate or incomplete semantic understanding (Moore, 1994; Tall & Vinner, 1981) or difficulty relating their semantic understanding to a syntactic definition or proof (Raman, 2003; Weber & Alcock, 2009). Additionally, students may use semantic reasoning as a substitute for syntactic proof (Harel & Sowder, 1998, 2007).

When students have such difficulties with semantic reasoning, syntactic reasoning can help them produce proofs even if they do not fully understand them. Understanding may then develop through students' reflection on how syntactic proofs relate to their semantic understanding of the concepts involved (Weber & Alcock, 2009). On the other hand, students' struggles with syntactic reasoning, such as use of imprecise or incomplete definitions (Harel & Sowder, 2009; Vinner, 1983) or failure to use definitions to structure proofs (Harel & Sowder, 2009; Moore, 1994) may limit their ability to construct and understand mathematical proofs.

Both semantic and syntactic reasoning are important in proving, and the affordances above suggest that students may come to understand proving and proof in one of two ways: by using semantic reasoning to build syntactic proofs, or by making sense of syntactic proofs through reflections on their semantic understanding (Weber & Alcock, 2009).

Method of Inquiry

The data in this paper come from a larger study that investigates students' proofs of statements involving relations and functions.

Participants

The participants were ten undergraduate students at a public university in Ohio enrolled in a transition to proof course. Six students were secondary mathematics education majors, and one each was a computer science, meteorology, mathematical statistics, and applied mathematics major. Although the course was intended for sophomore level students who had not taken a proof-based mathematics course, only one participant met these criteria. The other students were juniors and seniors with varied levels of experience with proof-based mathematics.

Course Structure

The transition to proof course was an inquiry-based learning course taught by the author of this paper. The topics in the course were: problem solving, logic, set theory, proof techniques, counting, induction, relations, orderings, functions, and cardinality. Students read about and completed ungraded pre-work on new topics before class. In class, they discussed the pre-work in small groups, followed by whole class discussions and ungraded student presentations. Students had graded post-work due weekly, which could be discussed with others, but write-ups were to be individual. In addition, there were four quizzes, a midterm, and a final exam in class.

Data

The data come from the assigned coursework in the transition to proof course. Although the students completed a variety of tasks involving surjective functions, this paper focuses specifically on the three tasks below involving composition and surjective functions. Overall, three weeks of class were spent on functions, with four days including study of surjective functions. Surjective functions were introduced the first day on functions with the following definition from Schumacher (2001):

> A function $f: A \to B$ is said to be onto if for each $b \in B$, there is at least one $a \in A$ for which $b = f(a)$. In other words, f is onto if the codomain and the range of f are the same set.

In this definition, I consider the first sentence the syntactic definition and the second sentence a semantic understanding of the definition. The next three days of class focused on injective and surjective functions, composition of functions, and their interactions. Students constructed examples and explored conjectures on the composition of functions with both finite and infinite domains satisfying varied combinations of injective and surjective. The tasks examined in this paper were explored as pre-work and in class before being assigned as post-work and on in-class assessments, but complete solutions were not provided.

Task 1. Task 1 was on a post-work assignment due on the fifth day of study of functions.

True or False? If true, prove it; if false, provide a counterexample.

Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be functions. If the composite function $g \circ f: A \rightarrow C$ is onto, then g is onto C.

Task 2. Task 2 was on an in-class quiz, four class days after the due date for the post-work containing Task 1.

Let $A, B,$ and C be nonempty sets and $f: A \rightarrow B$ and $g: B \rightarrow C$ be onto functions. State the domain and codomain of $g \circ f$. Prove that $g \circ f$ is onto its codomain.

Task 3. Task 3 was on the final exam, four class days after the quiz containing Task 2.

True or False? If true, prove it; if false, provide a counterexample.

Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be functions. If the composite function $g \circ f: A \rightarrow C$ is onto, then f is onto B.

Results

Task 1

Every student correctly identified the statement in task 1 as true. Seven of the ten students used an indirect proof, but it was often unclear whether they were using proof by contradiction or contrapositive. Not a single student used the word "contradiction," and each indirect argument concluded $g \circ f$ was not onto, many starting similarly to "if g is not onto, then $g \circ f$ is not onto because…." Most subsequent arguments were based on semantic understandings of surjective functions instead of the syntactic definition. It is unclear what provoked the use of an indirect proof strategy, but it aligned almost naturally with their semantic reasoning in this context.

The students demonstrated five different semantic understandings of surjective functions, some specifically in the context of composition, and some which overlapped. Two students' arguments included diagrams such as those discussed in Task 3 below. Non-surjective functions in the diagrams are represented with an element in the codomain that is not in the range. Additionally, two students' arguments expressed this idea in words, specifically speaking of mapping elements:

> Assume g is not onto. If f is onto B, then all the elements in B can be mapped back to A. When we map B to C, not all of the elements of C can be mapped back to B. Since B is not onto C, A cannot be onto C. Therefore g must be onto C.

Another student argued similarly to the students above in the language of codomain and range:

> If $g \circ f$ is onto, is g onto? True. This is true because if $g: B \rightarrow C$ had an element show up in its codomain that was not in the range, then the mapping from $A \rightarrow C$ would contain that same element in its codomain and not its range.

Two students argued on the consequences of g be the last function applied in the composition:

> Suppose not, that g is not onto C. Therefore $g \circ f$ would also <u>not</u> be onto C. This is because g is the highest level function that provides the final range of the entire composite, and if g can't reach all of C, then the composite $g(f(x))$ certainly won't either.

Finally, three students used the idea that g and $g \circ f$ have the same range when $g \circ f$ is onto:

> True because the range of g will also be the range of $g \circ f$. So, if $g \circ f$ is onto, then that means f has its domain and range, the domain of g that has the same elements as range

of f will have a range also, and that range of g of those elements will be the same range of $g \circ f$.

Only two students used the syntactic definition of surjective on task 1. Each student gave a correct proof, with one being a direct proof and the other a proof by contrapositive.

Task 2

Nine of the ten students used a direct proof strategy on Task 2, and six students attempted to use the syntactic definition of surjective function. This approach was in stark contrast to students' approach to Task 1. However, in attempting to follow the forward structure of a prototypical direct proof – start with the assumptions and use definitions to work to the conclusion – students missed the backward structure of the definition of surjective and were unable to use it appropriately to structure their proofs. Each proof attempt started in the domain of $g \circ f$ and moved toward the codomain as in this example:

Let $a \in A, b \in B,$ and $c \in C$. Note $(g \circ f)(x) = g(f(x))$. Since f is onto, $\forall b \in B$, there is an $a \in A$ such that $f(a) = b$. Furthermore, since g is onto, $\forall c \in C, \exists b \in B$ such that $c = g(b)$. Suppose $(g \circ f)(a)$. $(g \circ f)(a) = g\big(f(a)\big) = g(b) = c$. Hence, $g \circ f$ is onto.

For an analysis of the difficulties that lead to this type of proof, see Epp (2009).

Four of the students who used a version of the syntactic definition also used semantic reasoning in their proof attempt, as illustrated in the following example:

Let $a \in A$. Since f is onto, $\exists b \in B$ such that $\forall a \in A, f(a) = b$. Every value in B is mapped to. Similarly with g, $\forall c \in C, \exists b$ such that $g(b) = c$. And since all values in B are mapped to, and g is also onto, all values in C get mapped to. $g \circ f$ is onto C.

Finally, four students used semantic arguments only – three based on all elements in the codomain of surjective functions getting mapped to, and one cardinality argument presumably based incorrectly on surjective functions having the same codomain and range.

Task 3

Every student correctly identified the statement in Task 3 as false and attempted to construct a counterexample using a diagram to represent the sets and functions as in Figure 1:

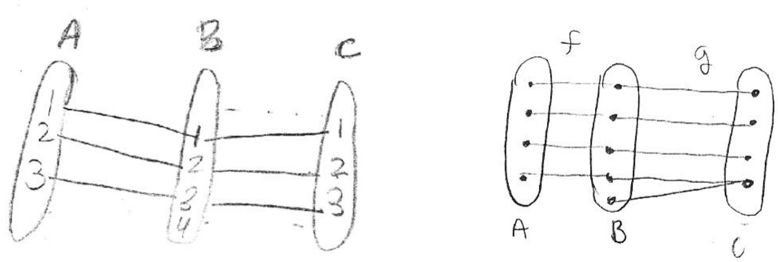

Figure 1. Sample counterexamples for Task 3

Three students provided a diagram only, although two of these students circled the element in B that was in the codomain of f but not the range. Four students accompanied their diagram with some version of the statement "$g \circ f$ is onto, but f is not onto." The other three students included explanations with their diagrams. One student used the syntactic definition of surjective and reasoned semantically about mapping elements in their explanation:

> Assume $g \circ f$ is onto, this means for each $c \in C$, there is at least one $a \in A$ for which $c = g \circ f(a)$. This means each c must map to a $b \in B$ so c can map to a. But there does not need to be an $a \in A$ for which $b = f(a)$ for every b as long as there is a path from $c \in C$ to $a \in A$.

This student's diagram was similar to the diagram on the right in Figure 1. The other two students used only semantic reasoning with their diagrams, with one student arguing that there was an element in B that was "unmapped by any element in A" and the other student using an argument based on the cardinality of B being greater than the cardinality of both A and C. Although each student's counterexample correctly showed that $g \circ f$ was onto and that f was not onto, f and/or g were not functions in half of the students' counterexamples, as is shown in the example on the right in Figure 1.

Discussion

The discussion will focus on two promising results: (1) Most students exhibited valid and useful semantic understandings of surjective functions (2) Some students tried to use their semantic understanding to make sense of the syntactic definition of surjective.

Semantic Understanding of Surjective Functions

Every student in this study demonstrated at least one semantic understanding of surjective functions, notably, some form of the diagram in Figure 1. Eight students exhibited at least one other semantic understanding. For the other two students, one used diagrams on every task, displaying no other semantic or syntactic understandings, and the other used the syntactic definition on Tasks 1 and 2. Overall, the students' semantic reasoning about surjective functions was correct and useful in arguing for the truth of statements and constructing counterexamples.

In addition to the diagram the students exhibited the following semantic understandings of surjective functions: having the same codomain and range, and all elements in the codomain being mapped to. Additionally, students easily negated these concepts for semantic understandings of non-surjective functions: diagrams with unmapped elements, elements in the codomain not being mapped to, and unequal codomains and ranges. Although these are simply different representations of the same concept, only one student demonstrated all three semantic understandings. It would be interesting to see if students could recognize and articulate the connections between these semantic understandings.

Finally, the students reasoned semantically about surjective functions specifically in the context of composition, including: considering the impact of which function was applied last in the composition; using the fact that g and $g \circ f$ have the same codomain; using diagrams representing both surjective functions and composition; and arguing using cardinality as mentioned above in Task 3.

Connecting Semantic and Syntactic Reasoning

On Tasks 1 and 3, only one student used both semantic and syntactic reasoning about surjective functions, but most students demonstrated useful semantic understandings on which they could build. However, on Task 2, as six students struggled to use the syntactic definition of surjective, four of them included semantic reasoning in their proofs to try to make sense of and connect to the syntactic definition. With more time and practice, these students' semantic understandings have the potential to be valuable in helping them understand the syntactic definition and structure of proofs of surjective functions.

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