When “Negation” Impedes Argumentation: The Case of Dawn

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Abstract: This study investigates one student’s meanings for negations of various mathematical statements. The student, from a Transition-to-Proof course, participated in two clinical interviews in which she was asked to negate statements with one quantifier or logical connective. Then, the student was asked to negate statements with a combination of quantifiers and logical connectives. Lastly, the student was presented with several complex mathematical statements from Calculus and was asked to determine if these statements were true or false on a case-by-case basis using a series of graphs. The results reveal that the student used the same rule for negation in both simple and complex mathematical statements when she was asked to negate each statement. However, when the student was asked to determine if statements were true or false, she relied on her meaning for the mathematical statement and formed a mathematically convincing argument.

Key words: Negation, Argumentation, Complex Mathematical Statements, Calculus, Transition-to-Proof

Many studies have noted that students often interpret logical connectives (such as *and* and *or*) and quantifiers (such as *for all* and *there exists*) in mathematical statements in ways contrary to mathematical convention (Case, 2015; Dawkins & Cook, 2017; Dawkins & Roh, 2016; Dubinsky & Yiparki, 2000; Epp,1999, 2003; Selden & Selden, 1995; Shipman, 2013, Tall, 1990). Recently, researchers have also called for attention to the logical structures found within Calculus theorems and definitions (Case, 2015; Sellers, Roh, & David, 2017) because students must reason with these logical components in order to verify or refute mathematical claims. However, Calculus textbooks do not discuss the distinctions among different connectives nor do they have a focus on the meaning of quantifiers or logical structure in algebraic expressions, formulas, and equations, even though these components are used in definitions and problem sets (Bittinger, 1996; Larson, 1998; Stewart, 2003).

Undergraduate students frequently evaluate the validity of mathematical conjectures that are written as complex mathematical statements. By *complex mathematical statements*, I mean statements that have two or more quantifiers and/or logical connectives. Other work has investigated students’ understanding of complex mathematical statements (Zandieh, Roh, & Knapp, 2014; Sellers, Roh, & David, 2017). However, these studies focus on students’ understandings of statements as written, and do not address students’ meanings for the negation of complex mathematical statements. Several studies have investigated student meanings for negation (Barnard, 1995; Dubinsky, 1988; Lin et al., 2003), but these studies do not explicitly address complex mathematical statements from Calculus. In order for students to properly justify why Calculus statements are true or false, and for students to develop logical proofs, they must understand a statement in both its written form and its opposite (Barnard, 1995; Epp 2003). For example, students at the Calculus level are asked to determine if sequences are convergent *or* divergent, if functions or sequences are bounded *or* unbounded. Thus, we also must explore student meanings for negation in the Calculus context—both the negation of an entire statement, and the negation of its logical components. In this paper, I will investigate one student’s meanings for the negation of various types of mathematical statements as well as how these negation meanings affected her justifications for several Calculus statements. Thus, I seek to investigate the following research questions for this student:

1. *As mathematical statements become increasingly complex, will a student keep the same negation meanings? If some or all of her negation meanings change, which meanings change and how do they change?*
2. *How do the student's meanings for negation affect her evaluations of complex mathematical statements from Calculus and her justification for these truth-values?*

Literature Review & Theoretical Perspective

Both colloquialisms for quantifiers and logical connectives as well as mathematical content may affect students' logic in mathematics courses. If I claim, "Every book on the shelf is French," the statement may be viewed as false if there are no books on the shelf. However, in mathematics, this statement would be vacuously true if there were no books on the shelf (Epp, 2003). If I claim "I'll get Chinese or Italian for dinner" one would assume that I was going to *either* get Chinese or Italian, but not both. We often use an *exclusive or* in our use of the English language, but in mathematics, we would consider that this statement would be true if both propositions were true (Dawkins & Cook, 2017; Epp, 2003). Students may also change their use of mathematical logic depending on the content of a mathematical statement (& Cook, 2017; Durand-Gurrier, 2003). Dawkins & Cook (2017) presented students with the statements "Given an integer number *x*, *x* is even or odd" and "The integer 15 is even or odd." Some students claimed that the first statement is true, but the second statement is false because they already knew that 15 is an odd integer.

Even if students correctly interpret a mathematical statement, their negation of parts or all of a mathematical statement may follow different conventions. If a statement contains more than one quantifier, students often negate only one of these quantifiers (Barnard, 1995; Dubinsky, 1988). Students may also leave disjunctions or conjunctions alone in a negation (Epp 2003; Macbeth et al., 2013). For example, some students negated statements of the form $P \wedge Q$ as $\sim P \wedge \sim Q$. In general, for all negations, Dubinsky (1988) claims that students often use *negation by rules*. The rules they use may or may not be correct rules of negation.

There may be other student meanings for negation that have yet to be discovered. My goal in this study is to describe my best perception of one student's own meanings for negations of complex mathematical statements at different moments. I use the phrase "student meaning" throughout this paper the same way in which Piaget views that each individual constructs his own meanings by assimilation and accommodation to schemes (Thompson, 2013). A *scheme* is a mental structure that "organize[s] actions, operations, images, or other schemes" (Thompson et al., 2014, p. 11). I cannot see a student's schemes, but can only do my best to create a model of students' negation schemes by attending to their words and actions throughout the clinical interview process. Schemes are tools for reasoning that have been built in the mind of the student over time. If a student repeats the same type of reasoning repeatedly, they begin to construct their *negation scheme* until the scheme is internally consistent. If students face inconsistencies, then they may adapt, or accommodate their schemes.

Some student meanings may be stable, but other meanings may be "meaning(s) in the moment" (Thompson et al., 2014). Thompson et al. (ibid) describe a meaning in the moment as "the space of implications existing at the moment of understanding" (p. 13), so students could be assimilating information in the moment by making accommodations to their current schemes. A student's thoughts may begin to emerge or different meanings may be elicited in different moments. Thus, I consider several different moments of interaction for each student because different moments of interaction may result in different types of student negation.

Methods

This study is part of a larger study that will seek to answer these research questions with undergraduate students from various mathematical levels. For this particular study, I conducted clinical interviews (Clement, 2000) with one student, Dawn, who is currently enrolled in a Transition-to-Proof (T2P) course. Dawn completed two clinical interviews that were each two hours long. Both interviews were video-recorded. One camera was used to zoom in on her work, while the second camera was zoomed out to capture her gestures. Different levels of tasks were chosen to determine if Dawn's negations stayed the same or changed across different levels of complexity. Clinical interview questions were used that would help me to determine why Dawn's negations stayed the same or changed across different tasks.

Interview tasks. I first presented Dawn with thirteen statements with one quantifier or logical connective to address my first research question. Two of these statements are shown in Figure 1 (left).

Statements with One Logical Component	Statement with Two Logical Components
1. Every integer is a real number. 2. 12 is even and 12 is prime.	There exists a real number b such that b is odd and negative.

Figure 1. Selected items with either one logical component or two logical components.

Dawn was asked to evaluate (i.e. provide a truth-value) and negate each statement, as well as evaluate her negations. After she completed these tasks, I presented her with a list of "other students' negations." I created these hypothetical negations based on variations of changing different parts of each statement. These hypothetical negations allowed me to test a wider range of possible negations that Dawn might accept as valid negations.

I also conducted a follow-up interview with Dawn to compare her negations of one logical component with her negations of complex mathematical statements in an attempt to begin to answer the second research question. I first presented Dawn with two statements, like the one shown in Figure 1 (right), which involves two logical components (an existential quantifier and either a conjunction or disjunction). I asked Dawn to evaluate and negate these statements in the same manner as she did in the first interview.

I later compared Dawn's negation of the more complex statements with her negations for the statements with one logical component to try to answer my first research question. I presented Dawn with three complex mathematical statements from Calculus, one of which is shown in Figure 2, in order to address my second research question.

There exists a c in [-1, 8.5], such that for all x in [-1, 8.5], $f(c) \geq f(x)$ and there exists a d in [-1, 8.5], such that for all z in [-1, 8.5], $f(d) \leq f(z)$.

Figure 2. Complex statement from Calculus.

I asked Dawn to evaluate if the statement in Figure 2 was true or false for seven different graphs. The statement given in Figure 2 is based on the conclusion of the Extreme Value Theorem (EVT) and the intervals shown in the graphs. Some of these graphs had only one of either an absolute maximum or absolute minimum, some graphs had neither an absolute maximum nor an absolute minimum, and some graphs had both an absolute maximum and an absolute minimum.

These graphs were selected in hopes that Dawn's data would include some moments where I could explore Dawn's negations in the context of her justifications for statements that were false (in her opinion). The Extreme Value Theorem (EVT) only holds for continuous functions. Since I omitted the hypothesis of the EVT, there are cases where this statement I present is false. I was then able to compare the negations that were part of her justifications with her previous negations with other statements.

Data analysis. My analysis was conducted in the spirit of grounded theory (Strauss & Corbin, 1998) using videos of the student interviews as well as the students' written work. Hence, the consistencies and inconsistencies in Dawn's negation meanings emerged from the data. I identified moments where distinctions could be made about Dawn's negation meanings. ew moment began when Dawn was presented with a new question or task, she changed her evaluation or interpretation of a given statement, or if she changed her argument or negation of a statement in any way. After identifying these moments of interest, I compared Dawn's one-component negations with the two-component negations. Finally, I compared her negations in the context of her justification for the Calculus statement with all previous negations.

Results

A consistent pattern emerged from Dawn's negations when I directly asked her to provide negations. However, when I presented graphs and asked Dawn to evaluate the validity of the mathematical statements for each graph, Dawn's negations in her argumentation did not always match her previous patterns of negation. A difference in interview questions appeared to influence Dawn's patterns for the negation of logical connectives and quantifiers.

Consistencies Across Negations

Dawn stated that in order to determine a valid negation, she could negate one part of the original statement, but not both parts of the original statement. I asked Dawn to explain why she believed she should only change one side of a statement for its negation. She stated, "In general, it's just some kind of rule that I follow, like you only negate one side." She also stated that negations for the same statement could have a variety of different truth-values (i.e. for the same statement, one negation that she deemed valid could be true while another negation that she deemed valid could be false). Since Dawn relied on a negation by rule and accepted negations with various truth-values, the evidence suggests that her overall meaning for the word "negation" was related to a constructed procedure rather than a statement that could prove or disprove the original statement.

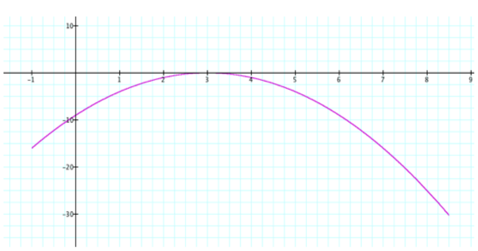
I first noticed Dawn's use of this procedure for statements of the form $\exists x, P(x)$. For statements with an existential quantifier of the form "There exists an x such that $P(x)$," she referred to "There exists x" as one part and "such that $P(x)$" as another part of the statement, and claimed that she "could only negate one part of the statement." Dawn would not accept negations of the form $\forall x, \sim P(x)$. She said that changing the "there exists" to "for all" would be "changing too much." Dawn usually preferred to start with the negation of the form "There does not exist an x such that $P(x)$," which is a valid negation. However, she also stated that statements of the form "There exists an x such that *not* $P(x)$" were valid negations. For example, for the statement, "There exists a whole number that is negative," Dawn wrote both "There does not exist a whole number that is negative" and "There exists a whole number that is not negative" as negations.

Dawn's algorithm for negating one part of a statement was also consistent with her negation of statements with a conjunction or disjunction because she still claimed that she could negate one part of a statement, but not both parts of the statement. For the statement, "12 is even and 12 is prime," Dawn wrote the negations, "12 is odd and 12 is prime" and "12 is even and 12 is not prime." For both of these negations, Dawn retained the logical connective and changed one part of the original statement in each negation. (Dawn usually kept the disjunction or conjunction from the original statement in her negations, but she sometimes accepted hypothetical negations that altered the logical connective if she felt as though a negation had the same meaning as the original statement.)

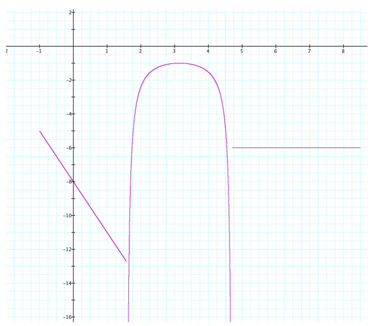
A combination of negation meanings. The statement "There exists a real number b such that b is odd and negative," has two logical components. Dawn interpreted the negation of both the quantifier and the conjunction in this statement in a similar manner as her earlier negations, as seen in her two negations: "There does not exist a real number b such that b is odd and negative" and "There exists a real number b such that b is even and negative." These negations are similar to the negations she preferred for "there exists" statements in the first interview, as they are also of the form "There does not exist an x such that $P(x)$," and "There exists an x such that *not* $P(x)$" (even though her negation of $P(x)$ is incorrect). Yet again, she did not consider the use of a universal quantifier in her negations and only changed one part of the statement. Dawn also negated the proposition within the statement that contained a conjunction in the same manner that she did with the first set of statements. The phrase "b is odd and negative" has its own parts that Dawn also considered. She negated "b is odd and negative" as "b is even and negative." She verbalized that she could have also used "b is odd and positive" for this part of her second negation. I asked her to consider explaining to a friend why her negation for the first complex statement was valid, to which she replied, "I would tell them that [my negation is correct] because I changed the second half of the statement." This reply indicates that Dawn is assessing the validity of her negation on her rule for negating one part of the statement, rather than comparing the *meaning* of the negation with the original statement.

Negation in Argumentation: When Negation Isn't Viewed as Negation

In the previous examples, I detailed Dawn's treatment of negation when I asked her to provide a negation. In the last set of tasks, I did not ask her to negate, but rather asked her only to determine if the statements were true or false on a case-by-case basis and to justify her claims. For the statement shown in Figure 2, Dawn interpreted the original statement as intended. Dawn explained why the statement shown below is true for the given graph:

Statement & Graph Presented	Transcript
There exists a c in [-1, 8.5], such that for all x in [-1, 8.5], $f(c) \geq f(x)$ and there exists a d in [-1, 8.5], such that for all z in [-1, 8.5], $f(d) \leq f(z)$. 	D: There is a maximum y-value at 3 [x=3] and a minimum y-value here (points to (8.5, f(8.5)) . So no matter what x is, this [f(8.5)] is going to be the least y-value. I: So what part of the statement tells you [that] you need to focus on the least y-value and the largest y-value? D: Because we want to pick values for c and d strategically so that they are going to be the maximum and minimum y-value. I: What part tells us we're going to pick the max and min? D: Here, for all x, you want it to, no matter what the value of x, the value of $f(x)$ is going to change. And you want this statement here, this inequality, to hold true, and there's only one instance where that can be true—at the max or min.

Dawn appeared to have a conventional interpretation for this statement. She expressed that she needed to choose *the* maximum or minimum that works *for all x*. Dawn's meaning for this statement and its negation was also revealed in her explanation of when the statement is not true. In the following example, Dawn claimed that the same statement is false for this case.

Statement & Graph Presented	**Transcript**
There exists a c in [-1, 8.5], such that for all x in [-1, 8.5], $f(c) \geq f(x)$ and there exists a d in [-1, 8.5], such that for all z in [-1, 8.5], $f(d) \leq f(z)$. 	D: The minimum y-value is $^-\infty$, so you couldn't pick a value for… d, that would always make this inequality true (points to $f(d) \leq f(z)$). I: Let's say your friend said, "For all the values that I look at, for all the y-values that I look at, if I chose *any* value for d, then I can always find a smaller value..." Would you agree with your friend's argument? D: Yeah, I would agree with his argument. I: Would you say that his argument is the same as your argument? D: Yeah, because I said there isn't a value for d, where there's the smallest y-value. I think that's kind of the same thing. It isn't the smallest because you could always find one smaller.

Dawn said that she *could not pick* a value for d such that this value of d would *always* satisfy the inequality. This response is similar to the negation "there *does not exist a d* such that $f(d) \leq f(z)$." Dawn's response was consistent with her prior approach to negate one part of a statement in her negation. Also recall that Dawn stated in the first interview that changing the second part of a statement and adding a universal quantifier would "change too much." Thus, I responded by asking Dawn to consider an alternative negation that used a universal quantifier and changed more than one part of the statement.

In the context of this statement where Dawn was asked about her argument rather than for a negation specifically, she accepted a negation that involved changing more than one part of a statement and she did not mention having an issue with the universal quantifier changing too much of the statement. Her original denial aligns with the argument, "there *does not exist* an x with a corresponding minimum y-value," but she also recognized that my proposed argument, "for *any* x-value, a smaller y-value than $f(x)$ can be found," was equivalent to her original denial. Thus, she accepted the argument that aligned with the negation "for *any* value of *d,* there exists a z such that $f(z) \leq f(d)$" by stating that this argument was "kind of the same thing" as her argument. She even explained why the logic for the two negations is equivalent: the y-value "isn't the smallest because you could always find [a y-value] smaller." Even though she had previously rejected alternate negations in the first interview that involved a universal quantifier, in the context of justification for this Calculus statement, she recognized that an alternate negation with a universal quantifier was valid.

In instances when my question or request omitting the word "negation," Dawn considered the *meaning* of the statement rather than her memorized rule to negate one part of the statement. Her interpretation of a statement and her negation for that statement varied based on the context of each mathematical statement. These moments in the second interview were characterized by the question, "Is this statement true or false for this graph?" rather than the command "Negate this statement." The word "negation" appeared to alert Dawn to negate only one part of the statement. However, when asked to think about the validity of a statement in a particular context, Dawn's approach was to use her reasoning to apply logical argument.

Conclusion & Discussion

When responding to negation tasks in the first interview, Dawn negated one part of a given statement, but not both parts of a given statement. This finding is similar to Dubinsky's (1988) finding that students tend to use rules (which may or may not be correct) to negate a statement. Dawn's meaning for the command to "negate" was to change any one part of the given statement, no matter what type of statement that was given. Her procedural approach for negating a statement could help explain why students only negate one of two quantifiers when statements contain multiple quantifiers (Barnard, 1995; Dubinsky, 1988) and why they often retain disjunctions and conjunctions in their negations (Epp, 2003; Macbeth et al., 2013). In Dawn's words, changing two quantifiers or changing a logical connective might be "changing too much" in the student's view. Dawn's negations are also consistent with other literature that has claimed that students' logic can change across different tasks (Dawkins & Cook, 2017; Durand-Gurrier, 2003). Evidence from this study indicates that we may have undergraduate students in our classes who only *negate by rules in certain mathematical contexts*. The negation scheme evoked with tasks that used the word "negation" did not appear to be elicited with tasks that did not use the word "negation."

For many students, the word "negation" may be associated with a *procedure* rather than using logical arguments based upon their own *reasoning*. Dawn applied the same negation meaning even as statements became increasingly complex as long as I asked her to negate. However, the command to determine if a statement was true or false actually led her to negate according to mathematical convention. Students who apply a memorized rule to negate a mathematical statement may have the ability to negate appropriately if the word "negation" does not hinder their argumentation. Students may benefit from tasks that use the command, "Explain why the following statement is true or false" and from negating quantifiers and logical connectives in different mathematical contexts. Then, the students may be asked questions that may help them construct their own rules for negation that are consistent with their argumentation. The word "negation" may be more appropriate to use *after* students have already used negation and constructed their own rules for negation based on their own reasoning about multiple mathematical statements.

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