Insights into Students' Images of a Geometric Object and its Formula from a Covariational Reasoning Perspective

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In covariational reasoning, when a student conceives of a situation as composed of measurable attributes that vary in tandem, discussing the relationship between quantities represented in a formula requires an interplay between a students' image of the situation and their conception of a formula. In this study, I categorize four pre-service teachers' images of both the situation and the formula as they describe the relationship between a given triangle's height and area. The results indicate how students' images of the situation and conceptions of a formula influence reasoning about the relationship between two quantities, specifically the role of numerical values and the development of a sophisticated dynamic image of the situation from which the student is able to draw conclusions.

Keywords: Covariational Reasoning, Cognition, Geometry, Pre-Service Secondary Teachers

Introduction

Researchers have identified the importance of covariational reasoning – conceiving of a situation as composed of measurable attributes that vary in tandem – in numerous K-12 topics including ideas surrounding rates of change (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Confrey & Smith, 1995; Ellis, 2007; Johnson, 2015; Moore, 2014; Thompson, 1994). In undergraduate mathematics courses, students often invoke ideas of rate of change when reasoning about formulas (e.g., using formulas for basic shapes to find the area under a curve, deriving formulas for areas and volumes using derivatives and antiderivatives). However, following the Common Core Standards Initiative (2010), students' middle and high school experiences with formulas mostly involve informal proofs using manipulations of static objects (e.g., using Cavalieri's Principle); these types of experiences do not give students the opportunity to explore how covariational reasoning can be invoked when reasoning about geometric objects. In other words, this treatment of formulas does not consider the variables involved in the formula as *varying* as described by Thompson and Carlson (2017). With these ideas in mind, I explore how four pre-service secondary teachers (heretofore referred to as students) who have successfully completed an undergraduate calculus sequence reason about the area of a geometric shape, a triangle, and its height. Thus, the research question for this report is the following: (How) do students explore the covariational relationship of the height and area of a well-known shape, an isosceles triangle, using a formula?

Background and Theoretical Perspective

In this study, I attend to students' images of a situation and students' conceptions of formulas when describing the relationships between quantities—measurable attributes of objects (Thompson, 1994). Throughout the study, I assumed the idiosyncrasy of individual's conceptions of quantities and images of a situation because I approach quantities as actively constructed by an individual (Steffe, 1991; von Glasersfeld, 1995). Moreover, I assumed when an individual constructed a relationship between quantities, the individual relied on their understanding of the quantity and their image of the relevant situation, which may have evolved over the course of the interview. Because of these two assumptions, I did not assume that students conceive the situations I provided to them in the same way I did; that is, they conceptualized the task using different quantitative structures. I noticed some of these differences, for example, when one student discussed changing the height of the triangle by drawing new triangles beside one another while other students described a smooth image of one triangle whose height was varying.

Moore & Carlson (2012) and Thompson & Carlson (2017) indicated how students' images of a situation impacted their reasoning, and other researchers have observed students reasoning quantitatively about real-world situations leading to their successful construction of equations (e.g., Ellis, 2007; Izsák, 2000; Moore & Carlson, 2012). Ellis (2007) differentiated this type of interplay between a student's image of a situation and their understanding of an equation in terms of covariational reasoning based on a student's attention to quantities. She, and other researchers who have prompted students to construct formulas via covariational reasoning using *area* situations, (Matthews & Ellis, In Press; Panorkou, 2016; Stevens et al., 2015) relied on the covariational reasoning defined by Carlson et al. (2002). In this view of covariation, giving the students a situation with which to operate is crucial. One popular context is that of a growing rectangle, which Thompson (1999) proposed and researchers have implemented with elementary and middle school students (Matthews & Ellis, In Press; Panorkou, 2016).

Alternatively, in the covariational reasoning described by Confrey and Smith (1994), students identify patterns in tables and use those numerical patterns to construct an equation. In this case, students are likely reasoning about values abstracted from operations of measurement rather than quantity's measurements (Thompson & Carlson, 2017). I use these two ways of covariational reasoning to differentiate students' reasoning with formula values versus those students who reason using their image of the situation.

I also situate my discussion around the results of two studies from researchers who have used growing area contexts. Matthews and Ellis (In Press) used a growing triangle context in their work. In their situation, the triangle's base remained on the left side of a square and the third vertex of the triangle began at the bottom left corner and traveled counterclockwise around the square. Although the two middle schoolers in their teaching experiment eventually successfully produced a normative graph of the distance the moving vertex traveled and the area of the triangle, never explicitly referencing a formula, the authors offer a caveat that the students may have reached their conclusion of a constant rate of change based on perceptual features, such as the constant speed of the traveling vertex. The authors also mentioned the difficulty of measuring area in their context. I extend this study by offering a point of comparison by working with students who have had vast experience with symbolization, area, and rates of change.

Stevens et al. (2015) also used an area context in a study with pre-service teachers; the pre-service teachers watch a dynamic image of a growing cone whose slant stayed at a constant angle and whose height grew and then shrank at a constant speed. The pre-service teachers described the relationship between the surface area and height of the cone. Half of the ten students in the study attempted to create and use a formula to determine the relationship between the quantities, and only two students constructed a graph using images of covariation rooted in the situation. None of these students produced a normative formula for the surface area of a cone. These results highlight the difficulty students seem to have relating a covariational relationship between two quantities they have constructed using their image of a situation with a formula that represents that relationship. It is important to note that in this task, both the 3-D nature of the cone and the formula for the surface area of the cone may have contributed to the students' difficulties with the task. This study extends this work by examining the relationship between the two when the student is given a simpler 2-D image and can produce a correct formula.

Methods

In an effort to focus this study on exploring ideas of how students connect ideas of rates of change and geometric objects, I chose a population of students who have had vast experiences with both. The four participants of the study were either in their first or second semester of secondary mathematics teacher program at a large public university in the southeastern U.S. Each student had completed a Calculus sequence and at least two other upper level mathematics courses (e.g., linear algebra, differential equations) with at least a C in the course. The students had all been enrolled in a spring semester content course exploring secondary mathematics topics through a quantitative and covariational reasoning lens using the *Pathways Curriculum* (Carlson, O'Bryan, Oehrtman, Moore, & Tallman, 2015). I interviewed all four students who expressed interest in the study after contacting the entire class about the study. This particular study focuses on the second task of a semi-structured clinical interview style (Clement, 2000) pre-interview in a series of 3-5 interviews; the interview was exploratory in nature. These pre-interviews lasted about two hours each, and for two of the interviews, there was an observer present. I encouraged the students to think aloud (Goldin, 2000) and attempted to ask only questions that would enable me to construct viable models of the *students' mathematics* (Steffe & Thompson, 2000).

Each interview was videotaped and these videos were digitized for analysis. Using an open (generative) and axial (convergent) approach (Strauss & Corbin, 1998), I offer distinguishing features of students' approaches to the *Growing Triangle* task based on my models of their images of the situation, their conception of their formula, and the role of the two in their description of a relationship between quantities. Students interpreted the relationship between the quantities differently; three of the four wanted to draw a conclusion about the directional change between quantities, and only one of the four attempted to consider amounts of change in one quantity with respect to the other. The results highlight further distinctions.

Task Design- *Growing Triangle*

In *Growing Triangle*, the student views a sketch on *Geometer's Sketchpad* of a static isosceles triangle. Students can drag a vertex of the triangle to increase or decrease its height and base while maintaining an isosceles triangular shape (i.e., $\overline{AB}$ stays constant but point C can be dragged along the perpendicular bisector of $\overline{AB}$) (Figure 1). Only Charlotte dragged C. I asked each student to "describe the relationship between the *height of the triangle* and the *area of the triangle*." All students considered the height to be the perpendicular distance from point C to $\overline{AB}$.

I purposefully designed the task to be a static image to see if asking about the relationship between two quantities would invoke a sense of change independent of watching a dynamic image. Also, not providing a dynamic image enabled me to avoid students concluding a constant rate of change based on a perceptual feature of the constant movement of a dynamic vertex. I also chose not to identify a specific height on the triangle for two reasons. First, I did not want to restrict their thinking if a student were to imagine rotating the figure at any point and wanted to consider a different height (Charlotte did, but quickly abandoned the idea). Second, by not identifying two particular quantities in the situation, I gained insight into what about the given situation they thought would stay constant and what would change in order to make a conclusion about the two given quantities. For instance, two students (Kimberley and Charlotte) considered different bases and categorizations of triangles (e.g., changing base with constant height, equilateral triangle shape maintained). For those students, I let them make a conclusion before directing them to consider specifically the case when $\overline{AB}$ stays constant and point C changes.

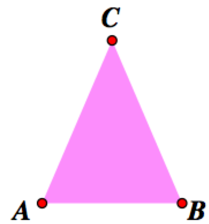


Figure 1. Original position of triangle ABC.

Results

I describe how each student's image of the situation and use of the formula played a role in how they described the relationship between the height and area of the triangle.

The Case of Jordan

Jordan was quick to discuss both a formula for the area of a triangle and to relate her conclusion to her image of the situation. When given the prompt, she said, "My first thought is the formula for the area of a triangle is one-half the base times the height, so if everything else is staying constant, except for height, which is increasing, then I would think the area would increase." Although her language may have seemed to indicate that she was reasoning about the formula, she says that she is imagining "*C* just like being pushed up," indicating that she was imagining the triangle varying as she was reasoning. Jordan's follow-up statement is further evidence she was reasoning by imagining at least one other image of a triangle:

> *Jordan*: So I'm thinking if it goes the other way. So, if you take C and you drop it [*motions finger downwards*], then it would decrease. Then there's just not as, like the base is staying the same, so [*makes pinching motion with fingers*]… if you have a squat triangle [*reaches to sketch and makes pinching motion smaller than the given height of the triangle*], like if I took *C* and dropped it, you can draw inside of this one [*motions where the two legs of the shorter triangle would be given the height of the squat triangle*] and see that it's taking up less space as this one [*makes circling motion around original triangle*].

After making this statement, Jordan justified her conclusion about her gross comparisons of the areas of the triangles by noticing that as *C* "goes up", the angles from the base and the legs increase, and said that the smaller triangle would fit inside the bigger triangles. After making this statement, she drew Figure 2 to illustrate her thinking. From this drawing, she concluded, "As *h* increases, *A* increases, and as *h* decreases, *A* decreases (see Figure 2)," once again supporting the idea that she imagined a dynamic image of a triangle. Moreover, her dynamic image of the triangle had quantitative entailments and invariant properties upon which she could operate in order to make a conclusion about the relationship between the two given quantities.

When asked if she could make any other conclusions about the relationship between the quantities, she stated, "I don't know by how much the area is changing when the height is changing." She added that she was not sure whether "it [would] be a constant change…I just can't picture like the height changing consistently, how that would change the area." This statement indicates that although her image of the situation had quantitative entailments and invariant properties sophisticated enough to imagine directional change of the area of the triangle with respect to height, it did not entail images of amounts of change in the quantities' magnitudes she could quantitatively compare to one another. I also note that Jordan did not return to her formula to attempt to make conclusions about how much the area is changing. In conclusion, Jordan relied only on her image of a dynamic situation to make conclusions about the relationship between quantities; she expressed knowledge of a formula for the area of a triangle, but she did not assimilate reasoning with the formula as a way to make conclusions about the rate of change between the height and area of the triangle.

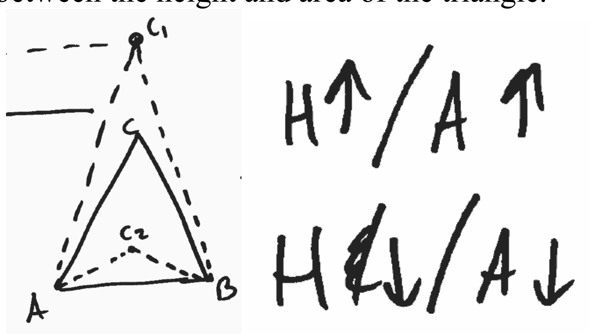
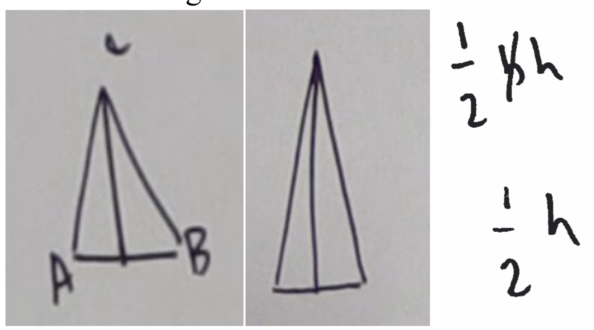


Figure 2. Jordan's illustration of the triangles resulting after moving point C up and down and her conclusions about the relationship between the height and area of the triangle.

The Case of Kimberley

Kimberley drew and considered a different instance of the triangle's growth when asked to consider the case when $\overline{AB}$ stays the same and C changes, but she was unsure how to make gross comparisons of the areas of the resulting triangles she drew. She drew a triangle (Figure 3b), noted that, in this case, the height has increased and said, "I'm not sure that the area has increased" when comparing it to the image on the screen (Figure 1a) "'cause we're getting [*makes narrowing motion with hands by bringing palms of hands closer to one another*], well [*pause*]". This statement indicated she was attempting to make a gross comparison of the area of the original triangle and the triangle with an increased height, but her image of the two triangles' areas, like Jordan's, did not afford her a way to make a conclusive comparison.

After this statement, she paused before saying, "Then the area of a triangle is one half base times the height [*writing* "½*bh" on her paper*]. So we keep-kept that part the same *[highlighting* $\overline{AB}$ *in her drawing of the original triangle* (Figure 3a)], that part [*highlighting bottom of triangle in Figure 3b*], but then the height increased [*motioning upwards from base of triangle*]." After making these comparisons between the bases and heights of the two triangles, she immediately concluded, "So then the area did increase." She justified her conclusion by saying, "So we know base is the same [*crosses out "b" in* "½*bh*" (Figure 3c)], so we can just look at what my height [*writes* "½*h"* (Figure 3c)], one half the height is, so if we know that the height is this here [*traces along height in original triangle* (Figure 3a)], but we know it got bigger here [*traces along height in Figure 3b triangle*], then that would have to be bigger [*pointing to* ½*h*] in this scenario." When asked to clarify how that discussion related to area, she said, "The area would be bigger because the height would be bigger." This discussion indicated a shift in how Kimberley was analyzing the situation. When her image of the situation did not afford her to make a conclusion about the relationship between the given quantities, she recalled and relied on a known formula for a triangle's area. Specifically, she compared her two cases and noticed that only h would have a different value in her formula. She did not use specific values, but rather unknown values such that one was greater than the other. By also noting the letter b in her formula was inconsequential to the resulting values for comparison because the base of the triangle stayed constant, she realized that by making a gross comparison of the values of ½h, she could also make gross comparisons of the area. Thus, she concluded that an increased height implied an increased area. In summary, Kimberley's conclusions about the areas of the two triangles resulted from a comparison of unknown values in a formula after a comparison of her image of the areas of the triangles was insufficient for her to make a definitive conclusion. She connected her conclusions back to the situation, but unlike Jordan, Kimberley's justifications relied on her conclusions from reasoning with the formula.



(a) *(b)* *(c)*

Figure 3. (a) Kimberley's representation of the original triangle given in the sketch with height labeled, (b) drawing in the case when C is "higher" than the initial point, and (c) reasoning with the formula for a triangle's area.

The Case of Charlotte

Unlike Kimberley, Charlotte immediately considered using a formula for the area of a triangle; when Charlotte is first presented with the task, she asked if it can be "something I've already been taught about triangles and areas and heights." Also unlike Kimberley, Charlotte was unsure about her recollected formula, stating that area equals "base times height" but that the formula may only be true for right triangles. Being unsure of her formula, she turned to reasoning with the static image of the triangle because, as she later reflected, "I tried to throw it [her formula] away, because I thought it only applied to right triangles for a moment" and that she did not know a formula for the triangle in front of her. She claimed that an increase in height implied an increase in area, noting that she was "picturing [her]self dragging this *C*". Thus, like Jordan, Charlotte had a mental image of a dynamic triangle in mind to try to make a conclusion about the relationship between the given quantities. However, her dynamic image was not as sophisticated as Jordan's image because Charlotte was unable to justify her claim as Jordan did. For instance, Charlotte dragged point *C* up in the sketch to illustrate to me how she was thinking of the situation, and remarked that "it made a bigger pink space" but later returned to the situation to drag *C* again and said, "If I increase and decrease that [height], wait, am I changing the area? Yea, definitely. At some points, it's easier to tell than others, but I feel like I'm changing the area." Like Kimberley, Charlotte had difficulty comparing areas with different heights, relying on what she *felt* was happening, rather than being able to draw conclusions using her image of the situation. Charlotte gave some insight into her image of the situation when we returned to this task later on in the interview when I asked her to draw how she was imagining "moving C up." She drew in Figure 4a and says, "I don't know. I can't draw that. Can I see it [the animation] again?". Upon doing so, she drew Figure 4b. From those drawings, we see an idea of narrowing sides (perhaps similar Kimberley's image) (Figure 4a) and a notion of stacking (Figure 4b), neither of which support an image of an isosceles triangle with increasing height that would enable her to identify amounts of change in area for equal changes in height.

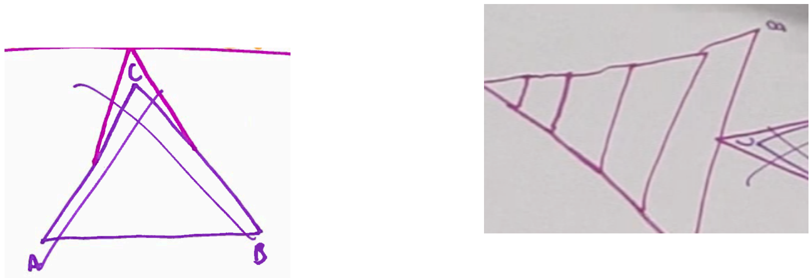


Figure 4. Charlotte's first (left) and second (right) attempt to draw her image of the dynamic triangle.

The Case of Alexandria

Like Charlotte and Jordan, Alexandria immediately referenced a formula and writes " ½*bh*=*A*". She represented the triangle on the sketch in a way similar to Kimberley's triangle (*Figure 3a*), pointing out the base ($\overline{AB}$), the height (*vertical line*), and the area (*region inside triangle*). However, unlike the other students, she did not consider changing the size or shape of the triangle. To her, the formula itself *was* the relationship between the triangle's height and area. Each variable in the formula represented an unknown value she could identify in the situation. Thus, she had no intellectual need to consider different values for height or area to plug into her formula to make a conclusion about covariational relationships. Her image of the situation remained static through the discussion.

Discussion

From these results, I argue that a student who assimilates a formula to a given situation will not necessarily assimilate a task asking to describe the directional covariational relationship between quantities by using numerical values. None of the students in this study did. Moreover, if a student attempts to reason about the relationship between two quantities by focusing on the situation instead, their image of the situation plays a crucial role in their ability to justify their conclusions about the relationships between quantities. For instance, only Jordan's image of the situation had quantitative entailments and invariant properties that enabled her to justify the directional relationship between the quantities by seeing that one instance of a triangle fit completely inside another. Conversely, Kimberley and Charlotte were only able to provide intuitions about changes in area based on their image of the situation. Kimberley was able to justify her claim by reasoning with her formula. She did not use specific values and so we cannot say she reasoned numerically as described by Confrey and Smith (1994), but she did illustrate a separation from reasoning about quantities' measurements in the situation in order to make gross comparisons between two hypothetical unknown values that had a specific relationship to one another. Afterwards, she reconnected her conclusion to the situation. Lastly, only Jordan made an attempt to reason using amounts of change, but her image of the situation was insufficient for her to make a definitive conclusion. These results call for a way to support and scaffold students' images of change quantitatively that they might be able to make conclusions about rates of change using amounts of change.

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