A Framework for Analyzing Written Curriculum from a Shape-Thinking and (Co)variational Reasoning Perspective

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This preliminary study provides a framework to analyze the extent and nature of (co)variational and quantitative reasoning in written curriculum. In order to test and refine our framework, we examined both the narratives and worked examples in calculus textbooks on lessons dealing with the topic of functions. We present examples from those textbooks to illustrate the categories of our framework. We conclude with questions concerning potential areas to improve our framework.

Keywords: Textbook Analysis, Quantitative Reasoning, Covariational Reasoning, Calculus

Over the past couple decades, researchers have studied students' quantitative and covariational reasoning – the cognitive activities in which students conceive of measurable attributes varying in tandem (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) – and have categorized specific forms of reasoning. They argue its importance to understanding numerous K-12 topics (Carlson et al., 2002), and yet "many popular U.S. textbooks do not emphasize or support students in conceptualizing quantities and viewing function formulas and graphs as representing how two varying quantities change together" (Thompson & Carlson, 2017, p. 457). Paoletti, Rahman, Vishnubhotla, Seventko, and Basu (in press) have started analyzing graphs used in STEM textbooks and journal articles. Thompson and Carlson (2017) and Mesa and Goldstein (2014) reported how secondary level precalculus textbooks addressed the conception of function and inverse trigonometric function, respectively, in their textbook reviews. However, we were unable to find any textbook analysis frameworks that attend to the degree to which textbook narratives and worked examples provide students with the opportunity to conceptualize quantities or reason (co)variationally.

In this report, we describe our attempt to create such a framework. To do so, we adapt two cognitive-focused categorizations – Moore and Thompson's (2015) shape thinking constructs for graphs and Thompson & Carlson's (2017) variational and covariational reasoning framework into categorizations appropriate for analyzing static curricular materials. In order to test and refine our framework, we analyzed calculus textbooks sections readily available to us. We specifically analyzed the introductory material to calculus textbooks (i.e., the pre-calculus topics the authors included) because it provides insights into the conceptions of graphs, functions, etc. the textbook authors believe are foundational for students to have before entering calculus. In this paper, we introduce the framework with specific examples from our analysis.

Background and Rationale

Two main sources – shape thinking constructs (Moore & Thompson, 2015) and the framework for variational and covariational reasoning (Thompson & Carlson, 2017) – informed our construction of a framework that enables users to analyze the narratives and worked examples when textbook authors explain, define, or use terms, expressions, formulas, and graphs. Firstly, as we describe in more detail when introducing the framework, we adapt the shape thinking construct to analyze the extent to which the narratives and worked examples provide students with opportunities to develop quantitative and covariational reasoning.

Secondly, Thompson and Carlson's (2017) frameworks for variational reasoning and covariational reasoning enabled us to distinguish between various levels of covariation in the narratives and examples provided in the curriculum. We also benefit from other research (i.e., Carlson & Oehrtman, 2005; Cooney & Wilson, 1993; Confrey & Smith, 1994) to construct our framework. The structure in which those researchers provided various levels of understanding functions (i.e., correspondence vs. process/covariation view of functions) was useful in providing a way to analyze algebraically and geometrically defined functions in the narratives to determine how they promote opportunities for students to understand and use functions as values of two variables or quantities covarying.

Researchers have demonstrated that textbooks have significant influence on student learning and teacher practice (Begle, 1973; Schmidt, McKnight, & Raizen, 1997; Kilpatrick, Swafford, & Findell, 2001; Stein et al., 2007; Valverde, Bianchi, & Wolfe, 2002). For example, Kilpatrick et al. (2001) stated that "what is actually taught in classrooms is strongly influenced by the available textbooks" (p. 36). In particular, Carlson, Oehrtman, and Engelke (2010) showed a positive influence of a curriculum (i.e., *Precalculus: Pathways to Calculus*) on students' productive understanding of functions. They reported that students who completed a curriculum focused on quantities and their covariation scored significantly higher on the *Precalculus Concept Assessment* at the end of the course than at the beginning. However, given the important role of textbooks in students' learning and classroom instruction, there is limited investigation regarding how (co)variational reasoning is promoted in textbooks. Hence, we decided to develop a framework to analyze curriculum materials in order to determine the extent and nature of (co)variational reasoning provided students in the narrative and worked examples.

Framework

We had two main categories in our framework: static and emergent. We illustrate each of these categories along with examples from five calculus textbooks we investigated.

Static

We use the term *static* to refer to any instances of narratives or worked examples that do not reference quantities and relationships among them in ways that entail those quantities[1] varying. For example, we code things as *static* when they entail instances that provide students images of variables and formulas based on perceptual associations among visual shape, analytic form, and perceptual features. *Static* instances encountered in the narratives and worked examples during our initial work fell into several categories (see Table 1).

Perceptual Associations. One category is what we call *perceptual associations*. This category has subcategories (i.e., form-name, form-shape, shape-name, and property-shape associations). *Form-name associations* involve perceptual associations between an analytic form and a function class terminology (e.g., linear, quadratic, or exponential). We adapted this particular category from Moore and Thompson's (2015) shape thinking construct to account for representations that were not graphs but still seemed associated with a particular form of an equation. For example, the following description of linear function was provided based on its analytic form without giving attention to an invariant relationship between the variables x and $f(x)$ that change together: A function of the form $f(x)=ax+b$ is called a linear function (Larson & Edwards, 2010, p. 24; Rogawski, 2012, p. 13). *Form-shape associations* involved perceptual

[1] We use quantities here to refer to both a quantity's magnitude and a quantity's value. We return to this idea in the discussion section.

associations between an analytic form and shape of graph. For example, Johnston & Matthews (2002, p. 20) describe “a nonvertical line in the Cartesian plane, or (x, y) plane, can be described by an equation of the form $y=mx+$b” with little or no attention to the coordinate system or axes’ scales and no attention to the invariant relationship between variables x and y as they vary. We also recorded instances as *form-shape associations* when the narratives provide a perceptual association between a change in a parameter in the analytic form and a change in the shape of graph. For example, Edwards and Penney (2014) stated “[the] *size* of the coefficient a in Eq. (9) [$y=ax^2$] determines the ‘width’ of the parabola; its *sign* determines the direction in which the parabola opens” (p. 18). *Shape-name associations* involve perceptual associations between the shapes of graphs (e.g., “line” or “curve up”) and a specific function class terminology or name of a mathematical object. For example, Steward (2008) stated, “When we say that y is a linear function of x, we mean that the graph of the function is a line” (p. 24). *Property-shape associations* involve perceptual associations between the shape of graph and a feature of the graph (e.g., slope). For example, Rogawski (2012) provided a pair of parallel lines on a Cartesian coordinate axis that were not scaled and labeled, and stated that “[l]ines of slopes m_1 and m_2 are parallel if and only if $m_1 = m_2$” (p. 14–15)—with no attention to changes in one variable with respect to changes in another variable by considering the axes’ scales or orientations.

Table 1. Framework for coding the extent and nature (co)variation provided in the narrative of a lesson

Static	**Emergent**	
	Variation	Covariation
Perceptual associations	Continuous	Continuous
• Form-Name Associations	Gross	Gross Coordination of Values
• Form-Shape Associations	Discrete	Coordination of Values
• Name-Shape Associations		
• Property-Shape Associations		
Variable as unknown		
Correspondence		

Variables as Unknown. A second category of *static*, named *variables as unknown*, involves presenting a variable as having a fixed unknown value of a quantity or being only a visual symbol that is not varying in the way Thompson and Carlson (2017) categorized as “no variation” and “variable as symbol.” One example in Stewart (2008) is to “express the cost [C] of materials as a function of the width of the base” of a rectangular storage container (p. 15). In the solution, they note w and $2w$ as “the width and length of the base, respectively, and h be the height” and “the area of the base is $(2w)(w)=2w^2$” without considering h or w as varying. Furthermore, they use these variables to write an equation for C, also indicating a treatment of C as an unknown variable whose unique value needs to be computed.

Correspondence. We used the code *correspondence*, the third category under static, when the narratives provide instances in which there exists an established static link among numbers in sets, but there is no consideration of either the covariation of variables or the dynamic relationship between number of sets (Cooney & Wilson, 1993; Vinner & Dreyfus, 1989). We also coded instances as correspondence when they simply provide a rule for students to calculate a unique value of a variable or quantity by using any given value of another variable or quantity (Confrey & Smith, 1994). For example, Edwards and Penney (2014) provide the following definition of a function: "A real-valued function [*originally bolded*] *f* defined on a set *D* of real numbers is a rule that assigns to each number in *x* in *D* exactly one real number, denoted by $f(x)$" (p. 2). This definition is common across all the textbooks we investigated. These definitions do not provide a *process* view of function of how input values covary with output values, emphasizing the change over a continuum of values (Carlson & Oehrtman, 2005).

Emergent

The other main category in our framework, named *emergent,* identifies the narratives and worked examples representing various levels of varying and covarying quantities or variables based on Thompson and Carlson's (2017) outline of levels of reasoning variationally and covariationally. We adjusted those levels to fit a textbook analysis and used them as our sub codes under *emergent* to determine the level of opportunities provided in a written curriculum for students to develop quantitative and covariational reasoning. We acknowledge that Thompson and Carlson included smooth and chunky distinction for both variational and covariational reasoning. For the purposes of this framework, however, we chose not distinguish between chunky and smooth continuous (co)variation. We made this decision because of the research (Castillo-Garsow, Johnson, & Moore, 2013) done to indicate that it is the *student* who conceives of a situation as either chunky or smooth, and in our preliminary analysis, we did not find a narrative or example that attempted to distinguish between the two.

Covariational Reasoning (Thompson & Carlson, 2017). The other part of our framework under *emergent* outlines the level of opportunities to develop covariational reasoning. *Gross coordination of values* involves representing two variables or quantities whose values increase or decrease together without mentioning the individual values of variables as varying together in the narratives. For example, "As the independent variable *x* changes, or varies, then so does the dependent variable *y*" (Edwards & Penny, 2014, p. 3). *Coordination of values* involves instances of coordinating the values of one variable or quantity with values of another by providing specific and discrete pairs of values without providing the opportunity for students to conceive two variables or quantities whose value varies together in between those pairs of values. For example, a narrative in Edwards & Penny's (2014) box problem wants students to "[s]tart by expressing the box's volume $V = f(x)$ as a function of its height *x*, and then use the method of repeated tabulation to find the maximum value V_{max}" (p. 12). Here, the textbook offers various values for *x* and asks for students to find their corresponding values in order to determine when the value of *V* will be maximum. *Continuous covariation* involves instances providing a simultaneous and continuous change in the values of two variables or quantities. We do not have an example of this category; however, the narrative presented as an example for coordination of values would have been an example of continuous covariation if each of the functional representations was linked to a motion in a dynamic geometry software with a slider for students to change the value of *x* and simultaneously see the corresponding changes in the values of *V* continuously.

Variational Reasoning (Thompson & Carlson, 2017). One part of our framework under *emergent* outlines the level of opportunities to develop variational reasoning (i.e., discrete, gross, continuous variation). *Discrete variation* involves presenting a variable or quantity in the narratives as taking specific values, but without providing the opportunity for students to conceive the variable or quantity whose value varies in between those specific values. *Gross variation* involves presenting a variable or quantity whose values increase or decrease without mentioning the specific values of the variable or quantity while increasing or decreasing in the narratives. C*ontinuous variation* involves presenting a variable or quantity whose values increase or decrease continuously. We do not provide examples from textbooks for each category of variational reasoning because they can be seen in the examples for covariational reasoning. For example, we can see discrete variation in Edwards and Penny's (2014) box problem when they ask students to change the values of *x* to find the maximum values of volume. Here, the textbook provides the variable *x* as having specific values but without considering how its value varies in between those specific values.

Discussion

The aforementioned research upon which we base our framework describes how students think about quantities. We recognize that the frameworks we chose to adapt were cognitive in nature, and we contend that students conceptualize the written materials differently. For example, as Thompson and Carlson (2017) pointed out, "A variable's variation comes from *a person thinking* [emphasis added], either concretely or abstractly, that the quantity whose value the *letter* [emphasis added] represents has a value that varies" (p. 425). In other words, we cannot know whether a student will interpret a variable provided in a written curriculum as varying, a letter that has a fixed value, or as a symbol. Nevertheless, the curriculum (including textbooks) students receive will influence how students think and learn. Thus, although there will invariably be discrepancies in the intended curriculum, the written curriculum, and what students interpret from the written curriculum, we argue certain narratives promote ways of reasoning that scaffold students in a way that supports reasoning covariationally. Hence, we are constructing a framework to determine which conceptualization students are likely to have based on what we see as evidence from written curricula.

In our framework, we did not include the attention to the distinction between quantities' values and quantities' magnitudes because we have not seen any instances from textbooks representing this distinction. We note that the most sophisticated version of quantitative and covariational reasoning includes explicit attention to such difference (Ellis, 2007; Thompson & Carlson, 2017). Even though textbooks provide opportunities for students to develop productive ways of thinking about quantities that were identified in this study, we found it unfortunate how little evidence we found of curriculum materials intentionally supporting student development of sophisticated quantitative and covariational reasoning schemas. This missing emphasis in written materials may be a partial explanation for why researchers have identified students having difficulties with ideas such as rate of change (Carlson et al., 2002)

To conclude this report, we identify some of the challenges we experienced in developing the framework. For instance, when coding the narratives and worked examples, we determined any instance the textbook seemed to promote (co)variational reasoning or to develop static meanings for quantities and variables. The units of analysis varied from phrases, to sentences, to whole paragraphs, to specific representations of mathematical objects (e.g., graphs, tables, etc.), but we would like to define parameters for our unit of analysis. We will provide examples of how our current unit of analysis influences how we code specific textbook examples and narratives. Lastly, we have been considering various methods of reporting and further analyzing the data (e.g., by textbook vs. by topic, international vs. national) and discussing affordances each offers.

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