The Emergence of a Prototype of a Contextualized Algorithm in a Graph Theory Task

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Research has described the necessity and dangers of prototypes in mathematical learning, without offering explanations for what makes prototypes appropriate or inappropriate, or indeed how prototypes emerge in the first place. We explore one part of the emergence of a prototype: how a feature of a concept's example becomes predominant in subsequent generated examples. We describe how three students developed what they regarded as four examples and one non-example of an algorithm suitable for a client with a contextualized graph theory problem. The students engaged in a 'patching process' that preserved an inappropriate feature of the initial example in the other examples that were generated. We argue that the development of appropriate prototypes may depend on the types of processes (like the 'patching process') that students use to abstract and preserve features of the concept examples.

Keywords: Concept, Prototype, Algorithm, Graph Theory.

Introduction

Prototypes—those examples of a concept that are said to be *popular* or *typical*— play a significant role in mathematics learning (Hershkowitz, 1989; Tall & Bakar, 1992). On the one hand, prototypes make formal and abstract concepts more accessible (Tall & Bakar, 1992). On the other hand, prototypes can become obstacles when their properties that are unnecessary from the perspective of the formal concept, are perceived as something that any concept example must have (Hershkowitz, 1989). For instance, studies show that students are more likely to classify objects as examples of a concept when they "seem closer" (e.g., visually) to the prototype(s) (e.g., Presmeg, 1992). Accordingly, Tall and Bakar (1992) suggest that educators should help students to develop prototypes that are "as appropriate as possible" (p. 13), thereby implying that some prototypes are more appropriate than others. We propose that the appropriateness of a prototype is not its innate quality, but something that depends on the situation in which it emerges and is used. The study reported in this short paper comes to contribute to classic research that identifies prototypes that students have already developed (e.g., Hershkowitz, 1989; Rosch, 1979; Tall & Bakar, 1992), by exploring how prototypes emerge in the first place.

Theoretical Underpinnings

How have prototypes been conceptualized in the literature?

Rosch (1973) introduced the term *prototypes* to refer to examples of a category that were more 'central' or 'popular' (among a group of people) than others. The notions of centrality and popularity arose from the observation that humans perceive examples of a category as not having equal status—an example's 'closeness' to the prototype(s) influences its status (Rosch & Mervis, 1975). Hershkowitz (1989), conducting research in learning geometry, made a similar claim to Rosch's: "All the concept examples are mathematically equivalent [...] they satisfy the concept definition, but they are different from one another visually and psychologically. There are super examples which tend to be much more popular than all others" (p. 63). Tall and Bakar (1992) also observed that students, when asked if an object is a function, tended to answer "yes" if the object resonated with their prototypes, and "no" otherwise. We observe three different usages of the term *prototype* in this classic research of Rosch (1973), Hershkowitz (1989) and Tall and Bakar (1992). First, a prototype may refer to a concrete example of a concept, for example, a robin seen flying outside is a prototype of a bird. Second, the "category of a robin", that is, a set containing the defining features of a robin, is a prototype of a bird. Third, a prototype of a bird is a set of features that are predominant among all birds.

We use the third sense of the term prototype in our research by looking at how features of one example of a concept come to appear in other examples students generate. Specifically, the definition of *prototype* that we follow in this paper is: an abstract representation (as opposed to a concrete example) that possesses the most predominant features of examples of a concept (note our definition aligns with the way it is used in Dean, 2003; Rosch & Mervis, 1975). By this definition, the emergence of a prototype is equivalent to the emergence of predominant features among examples of a concept.

Consequently, our research question is: what processes are involved when a particular feature of an example appears in other examples of a target concept? We explore this question by analyzing the work of three students who worked together on a contextualized graph theory task, in which they were asked to develop an algorithm to satisfy a client's needs. After establishing their first example of the target concept, the group used a 'patching process' to generate their other examples. This particular patching process preserved a feature of the initial example in all the other examples that the group generated. We propose that this patching process is one instance of the processes that students might be engaged in when they abstract and preserve the features of the concept examples that become predominant.

What do we know about how prototypes emerge?

Research provides several explanations to how prototypes emerge. One explanation offered in mathematics education literature points to the role of our visual-perceptual limitations (Hershkowitz, 1989). That is, features that we "see" frequently among the examples of a concept are the ones that emerge to form our prototypes (but these frequently seen features are not necessarily equivalent to the defining features of the concept). This aligns with a common explanation in the cognitive psychology literature whereby prototypes arise out of frequent use (Taylor, 2003): if an example is repeatedly activated with the concept, then the example becomes a prototype. But Rosch (1999) argued that the frequency explanation falls short in some cases (e.g., even though children see blue and black skies equally, they almost always draw a blue sky when asked to associate the sky with a color). Another explanation suggests that an example becomes a prototype if it bears properties that are most common among other popular examples. In this case, prototype status is granted to the example by already existing prototypes (Taylor, 2003).

In what sense is an 'algorithm' a 'mathematical concept'?

We are aware that our use of the terms *concept* and *prototype*, and indeed our characterization of an algorithm as a concept may seem unconventional (not prototypical), so we offer a conceptual argument to justify this usage. Vinner (2014) refers to a *concept* as a generalization of instances that share certain things in common. Thomas (2014) defines an *algorithm* as "a step-by-step set of instructions in logical order that enables a specific task to be accomplished." Under these two definitions, an algorithm can be viewed as a concept because it is a generalization of structured instances that enable solving a particular task. Furthermore, in our study, we do not look at students' notions of an algorithm as an abstract entity. Instead, we are interested in the contextualized algorithms that students develop, algorithms that are proposed for a particular client with specific needs that can be derived from a contextualized narrative provided by the task.

Method

Participants, Research Setting, and Data Collection

The participants in this study were three students—Chad, Gil and Lome (pseudonyms)—who were enrolled in a pre-degree mathematics course in a large New Zealand university, and knew each other well as friends. None of the students had studied graph theory before, and they were studying high school level algebra when they participated in our research. Chad, Gil and Lome were recruited as part of a larger research project (Yoon, Chin, Griffith Moala, Choy, 2017) that involves over fifty undergraduate, secondary, and post-secondary/pre-degree students, and which explores student-mathematizing in tasks that present discrete mathematics concepts in contextualized narratives. Chad, Gil and Lome worked together on four discrete mathematics tasks in four one-hour sessions over the course of three weeks. These four sessions took place outside of class time and course requirements, and were audio recorded and video recorded. The group worked in the presence of an interviewer (the second author), who answered clarification questions about the wording of the task, but did not offer any mathematical hints.

Task

We report on Chad, Gil and Lome's mathematical activity in the third discrete mathematics task they worked on: "The Jandals Problem" (Yoon, Griffith Moala, & Chin, 2016). The task begins with some warm up questions that familiarize students with diagrammatic representations of graphs (networks) within the context of friendship associations, where a node represents a person, and an edge between two nodes represents a friendship between two people. After the warm-up questions a scenario is posed: Xanthe, an American exchange student in New Zealand learns that the locals use the word *jandals* to refer to what she commonly calls *flip-flops*. Upon returning home, Xanthe wants to spread the word *jandals* throughout different networks of friends like the ones shown below in Figure 1 and Figure 2. Students are asked to:

> Create an algorithm (method) that Xanthe can use to figure out the first person whom she should share the word with first in each friendship network to ensure that the word gets passed on to everyone in the network as rapidly as possible. She assumes that a person will share the word with all of his/her friends on one day, and each of those friends will share it with their friends the next day. Ensure that your algorithm will work for any friendship network, not just the one given [Figure 1]. (Yoon, Griffith Moala, & Chin, 2016, p. 12)

Only Figure 1 was initially given to the students; Figure 2 was given to them at a later stage.

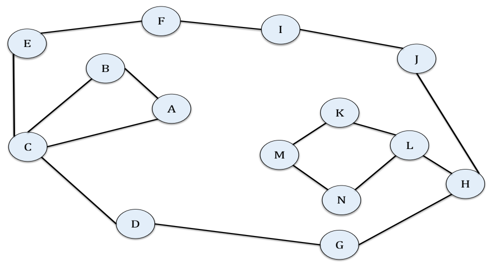

Figure 1. Friendship Network 1

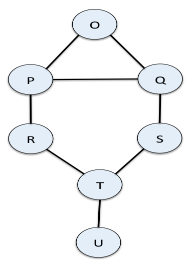

Figure 2. Friendship Network 2

This task asks students to develop "an algorithm (method) that Xanthe can use". Throughout the session, the students and interviewer switched between 'algorithm' and 'method', and we preserve both when describing and analyzing their work.

Data Analysis

Due to the exploratory nature of the study, the aim of the analysis was not to confirm existing constructs but rather to explore aspects of the data that may be used to construct plausible explanatory models (Clement, 2000) for how features of an example come to be predominant in subsequent generated examples. Thus, the analysis involved an "open interpretation of the data" (Clement, 2000, p. 548), which is "useful for constructing initial explanatory models of cognitive processes" (Koichu & Berman, 2005, p. 171) inferred from the data.

Following the task description, we regard the target concept that guides the students' mathematical activity to be "an algorithm (method) that Xanthe can use" with two defining properties: (1) it identifies the *quickest starting person*; (2) it must work for any friendship network. We searched the data for examples of the target concept that the students created, establishing the presence and predominance of a common feature among the examples. Then we followed the development of the examples (individually and collectively), looking for particular aspects of the group's work that may have contributed to the emergence of the predominant feature.

Findings

We present three episodes from Chad, Gil and Lome's activity during the Jandals problem in which they create an example of the target concept of "an algorithm (method) that Xanthe can use", and where a particular feature of this first example also appears in further examples that the students generate. Each episode begins with our *account of* (Mason, 2002) the group's work (i.e., addressing what happened) followed by our analysis (addressing *why* particular things happened).

Episode 1: A valid example of the target concept emerges

After the group reads the task instructions, Gil says they need to find the person in the friendship network (Figure 1) that would spread the word quickly. Lome suggests they choose a person, count how many days it would take for the word to spread starting from the chosen person, repeat the process for all other persons, then share the word with the person that gives the least number of days. Lome refers to this entire process as "the elimination method."

The students use the elimination method on the following persons in the first friendship network: *C, I, L, J, H, G, M*. They determine that the quickest of these is *H*, which yields four days, having incorrectly calculated that *G* yields six days, when in fact it also yields four days, making it another quickest starting person. Lome remarks, "I reckon we've solved it!" He then looks back at the task instructions, turns to the interviewer and says:

Lome: What's an algorithm? This [*points to written parts of their elimination method*] is not an algorithm is it?

Interviewer: An algorithm is like a method. So it's not your solution, it...

Gil: It's like the way you got it.

Interviewer: Yeah, so that she can use it for any other network, because this is just one of many different friendship networks across the campus.

Lome: Can we say we just did elimination method?

Interviewer: You've got to explain it as well as you can so that Xanthe can use it for a different one that she is given.

When Chad says he is still unsure what they need to produce, Lome says, "she needs to be able to figure out the solution to any network, from our method." Gil then suggests a method: "Yeah, so it would be like, your method would be like, the [*starting*] person should tell three people because [*points to Figure 1*] if you told *H*, *H* would tell *L*, *G*, and *J*. And then, it spreads." Lome and Chad both nod their heads, and Lome says "Yeah, cool!"

Analysis. Two different methods emerge for the group in this excerpt: *the elimination method,* which is the exhaustive search procedure that the group uses to find the quickest starting person in the first network; and *Gil's method* (share the word with someone who tells three people) which is the method Gil suggests giving to Xanthe. Lome's question to the interviewer, "Can we just say that we did elimination method?" can be interpreted as asking whether the elimination method qualifies as a valid example of the target concept. The group's subsequent decision not to share it with Xanthe suggests they do not consider it to be a valid example (although it is indeed a mathematically valid algorithm for Xanthe's purposes). On the other hand, the group's enthusiasm towards Gil's method, indicated by head nods and "yeah cool!" suggests they regard Gil's method to be a valid example of the target concept. Thus, although both methods are put forth as potential examples of the target concept, only one of them (Gil's method) is accepted by the group as a valid example of the target concept.

Episode 2: A feature of the first example is preserved in the generation of a second example and subsequent examples

After Gil proposes his method at the end of Episode 1, the interviewer points to the task instructions and says:

Interviewer: Can I get you to read what the method needs to do?

Lome: So [*looks at Figure 1*] she should share it with someone who tells at least three people. But then mind you, if she starts at *L*, *L* tells three people but it doesn't work as fast.

Gil: Yeah, that's true.

Lome: So maybe [*points to H*] the starting person needs to tell three people [*points to L, J, and G*] but one of those three people [*points to L*] has to tell two other people.

Gil: Yeah [*nods head*].

Lome: Because this person [*points to C*] tells four people, but none of those people [*C's friends*] are connected to two other people. That's a method. I'll write it.

Lome writes down: "Share the word with a person who tells three people, and one of those three people must tell two other people." The interviewer asks, "Are you happy?" Chad, Gil and Lome reply, "Yes."

Later in the task, Gil revises Lome's method to "share the word with someone who tells three people, and each of those three people must tell one other person." Then, Chad revises Gil's second method to "Share the word with someone who tells three people, and two of those three people must each tell one other person." These methods are presented in Table 1 below.

Analysis. After the group notices a flaw in Gil's method, all of the methods they subsequently suggest nonetheless preserve a feature of Gil's method: *the quickest starting person tells three people*. What may be a plausible explanation for the preservation of this feature? We propose that the group may have noticed that their method needed to perform two functions: (i) it had to find the quickest starting person(s), and (ii) it had to *not* find non-quickest starting persons in a given network. In light of these two functions, the process whereby Lome's method above is obtained by building on Gil's, can be described as: keep the part of the current method that satisfies the first function, and change the part that violates the first function (note that *'change'* also includes adding other parts to it) so that the second function is also satisfied. We refer to this process as a 'patching process" due to its change-only-what-needs-to-be-changed nature. We observed the students using this patching process to generate the other examples (see Table 1).

Episode 3: The elimination method is judged to be a non-example of the target concept

After Lome writes down his method—share the word with a person who tells three people, and one of those three people must tell two other people—the interviewer hands the group a sheet of paper on which is printed a new friendship network (see Figure 2 above), and asks them to show how Lome's method would work on this new network. Rather than apply Lome's on this new network, the group uses the original elimination method to find five solutions: *P*, *Q*, *R*, *S*, and *T,* which all give three days (note, their solution set is incorrect; *R* and *S* are the quickest starting persons as they only give two days). Then Lome says:

Lome: Can you get four days? [*Chad demonstrates that it takes four days starting from person U*]. OK, so why wouldn't she tell *U* but tell *Q* instead? What's the method? Obviously *Q* will be quicker but why would she tell *Q* and not *U*?

Gil: Because, read your thing [*points to Lome's method*]. She has to tell someone who tells three people, so *Q* tells three people. If you tell *Q* first, *Q* tells *S*, *P*, and *O*. Then, the second person must tell at least two other people.

Lome: Yeah.

The group again agree that they need to produce a set of instructions and a method that Xanthe can use on any network. Then Lome remarks:

Lome: Say she has hundreds of these [*networks*] she doesn't want to do elimination method for every one. What if there's a network with a thousand people? She'll be there for ages counting!

Analysis. In total, the group created five methods for the task, which are summarized in Table 1, four of which they regard as examples of an algorithm they could give Xanthe.

Table 1: The five methods created by Chad, Gil and Lome during the Jandals problem

Name	*Description of the method*	*Validity as example of target concept*
The elimination method	Choose a person, count how many days it would take for the word to spread starting from the chosen person, repeat the process for all other persons, then share the word with the person that gives the least number of days.	Non-example
Gil's method	Share the word with someone who tells three people.	Valid example until flaw is found
Lome's method	Share the word with someone who tells three people, and one of those three people tells two other people.	Valid example until flaw is found
Gil's second method	Share the word with someone who tells three people, and each of those three people tells one other person.	Valid example until flaw is found
Chad's method	Share the word with someone who tells three people, and two of those three people each tells one other person.	Final valid example

While Lome was comfortable using the elimination method to find person *Q* as one solution for the friendship network in Figure 2, his questioning of why Xanthe should choose *Q* over *U* suggests he did not regard the elimination method as adequate justification for this choice: "obviously *Q* will be quicker than *U*, but why"? Gil cites Lome's method to justify choosing *Q* over U, which seems to satisfy Lome. For Lome then, the elimination method was prescriptive without being explanatory. This perceived feature, together with his characterization of the elimination method as tedious for large numbers of people may have dissuaded the group from perceiving the elimination method as a valid example of the target concept, even though it is indeed a mathematically appropriate algorithm. Rather, the group perceived the elimination method as a non-example of the target concept, which, in giving the group an idea of what a valid example should (not) look like, may have contributed to inclusion of the feature "tell someone who knows three people" in the examples they generated afterwards.

Discussion and Concluding Remarks

The episodes that were presented in this paper provide an account of the recursive process that a group of students went through when engaging with a concept of algorithm. First, the group considered a method and decided whether it was an example or non-example of the targeted algorithm. The consideration was made against two functions that the group wanted their method to perform. Second, the group recognized that the method under consideration performed one of the functions, but not both. Lastly, a new method was generated in which feature from the previous method was preserved and a new feature was introduced so as to ensure that the resulting method performed both functions. We refer to this recursive process as 'a patching process' due to its change-only-what-needs-to-be-changed nature. This patching process eventuated in the preservation of a feature of the initial example in all the other examples that group generated, and hence the emergence of a predominant feature. We propose that this patching process is one instance of the processes that students might be engaged in when they abstract and preserve features of the concept examples—features that become predominant.

The patching process that we identified in our study puts forward the crucial role of the first concept examples that students encounter. This aspect aligns with the existing research on prototypes (e.g., Hershkowitz, 1989; Tall & Bakar, 1992). We, in our study, show that merely encountering examples is not necessarily sufficient, and that recognition of the example's status (as *an example of the* concept) is necessary. Indeed, the students in our study generated the elimination method, an algorithm that we, as researchers, wanted them to develop. Furthermore, to the best of our knowledge, there is currently no more efficient algorithm to cope with the tasks that were handed to our students. Yet, the group almost immediately labeled the elimination method as "not an algorithm," and its inappropriateness was not questioned in the data that we presented. Thus, it seems reasonable to propose that developing appropriate prototypes may come down to preserving appropriate features (and rejecting inappropriate features) of the examples; which in turn may depend on the processes (such as the patching process we found here) that students go through when abstracting and preserving certain features of the concept examples.

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