Teaching Linear Algebra: Modeling One Instructor's Decisions to Move between the Worlds of Mathematical Thinking

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In this article, we report results from a year-long study in which a linear algebra instructor worked with the research team to document his instructional decision-making via journals and interviews as well as to code and analyze the data. This work supports the development of a more general model of the instructor's decision-making and provides a lens with which to make sense of the instructors shifts between representations from each of Tall's Three Worlds. With the introduction of the model, we include an example to show how the various codes interact in the instructor's decision-making. We also provide a detailed description of one incident that provides a second perspective on the instructor's decisions, helping to support a more robust understanding of the data.

Keywords: Linear Algebra, Tall's Worlds, ROGs, decision-making

Theoretical background

Over the past decade, research on linear algebra has revealed that many students struggle to grasp the more theoretical aspects of linear algebra which are unavoidable features of the course and are focused on students' thought processes (e.g. Stewart & Thomas, 2009; Hannah, Stewart &Thomas, 2013; Britten & Henderson, 2009; Wawro, Zandieh, Sweeney, Larson, & Rasmussen, 2011; Gol Tabaghi & Sinclair, 2013; Salgado & Trigueros, 2015). The research in recent years have mainly concentrated on students' difficulties and with a few exceptions (Hannah, Stewart & Thomas, 2011; 2013; Zandieh, Wawro, & Rasmussen, 2017; Andrews-Larson, Wawro, & Zandieh 2017), research on instruction in linear algebra is still scarce.

Research in instruction at the university level is fairly new. As Dreyfus (1991) suggested, "one place to look for ideas on how to find ways to improve students' understandings is the mind of the working mathematician. Not much has been written on how mathematicians actually work" (p. 29). Two decades later, Speer, Smith, and Horvath (2010) declare that "very little research has focused directly on teaching practice and what teachers do and think daily, in class and out, as they perform their teaching work" (p. 111). In recent years some mathematics professors have been more willing to examine and reflect on their own teaching styles, leading to a growing body of research in this area (e.g. Paterson, Thomas, & Taylor, 2011; Hannah, Stewart, & Thomas, 2011). The overarching goal of this study was to contribute to this gap in the literature by examining a linear algebra instructor's thought process and teaching decisions over an entire semester.

The theoretical aspects of this study are based on Schoenfeld's (2010) Resources, Orientations and Goals (ROGs). He claims that "if you know enough about a teacher's knowledge, goals and beliefs, you can explain every decision that he or she makes, in the midst of teaching" (2012, p. 343). By resources Schoenfeld focuses mainly on knowledge, which he defines “as the information that he or she has potentially available to bring to bear in order to solve problems, achieve goals, or perform other such tasks” (2010, p. 25). Goals are defined simply as what the individual wants to achieve. The term orientations refer to a group of terms such as “dispositions, beliefs, values, tastes, and preferences” (2010, p. 29). Although, the theory was originally considered as applying to research on school teaching, (Aguirre & Speer, 2000; Thomas & Yoon, 2011; Törner, Rolke, Rösken, & Sririman, 2010), it clearly has applicability to research on university teaching (e.g. Hannah, Stewart & Thomas, 2011; Paterson, Thomas & Taylor, 2011).

As a part of the theoretical framework described in this paper, we also employed Tall’s three-world model of embodied, symbolic and formal worlds of mathematical thinking. Tall (2010) defines the worlds as follows: The embodied world is based on “our operation as biological creatures, with gestures that convey meaning, perception of objects that recognize properties and patterns...and other forms of figures and diagrams” (p. 22). Embodiment can also be perceived as giving body to an abstract idea. The symbolic world is the world of practicing sequences of actions which can be achieved effortlessly and accurately. The formal world “builds from lists of axioms expressed formally through sequences of theorems proved deductively with the intention of building a coherent formal knowledge structure” (p. 22). In Tall’s view (2013, p. 18), “formal mathematics is more powerful than the mathematics of embodiment and symbolism, which are constrained by the context in which the mathematics is used”. He believes that the formal mathematics is “future-proofed in the sense that any system met in the future that satisfies the definitions of a given axiomatic structure will also satisfy all the theorems proved in that structure. The formal mathematics can reveal new embodied and symbolic ways of interpreting mathematics.” (p.18).

We believe that in many cases teachers and text books move between worlds of mathematical thinking very naturally and rapidly, not allowing students time to discuss and interpret their validities. They assume that students will pick up their understandings along the way. As Dreyfus (1991, p. 32) declares “One needs the possibility to switch from one representation to another one, whenever the other one is more efficient for the next step one wants to take… Teaching and learning this process of switching is not easy because the structure is a very complex one.” We hypothesize that most students do not have the cognitive structure to perform the switch that is available to the expert. For example, Duval (2006) noted that to construct a graph, most students have no difficulties as they follow a certain rule “but one has only reverse the direction of the change of register to see this rule ceases to be operational and sufficient”. (p. 113)

In this study, we employed Tall’s three-world framework of embodied, symbolic and formal to follow a linear algebra instructor’s movements between the worlds. Our research questions are: How did the instructor’s ROGs inform his movements in the three worlds? When did he decide to move between the worlds and why?

Methods

This narrative qualitative study examined an instructor’s teaching journals. The study took place over an entire semester, during which the instructor (David) was teaching two sophomore-level linear algebra courses using the IOLA curriculum (Wawro et al, 2012). With some exceptions, the instructor kept a journal of teaching reflections throughout the semester and met with the research team (lead investigator, senior investigator, and undergraduate assistants) each week. The reflections and team meetings allowed for triangulation of data and gave multiple chances for the instructor to share his reasoning about his teaching decisions. The team then conducted a retrospective analysis of the journals following the methodology of narrative study (Creswell, 2013). Specifically, the team iteratively coded the data, beginning with open coding that each member of the team conducted and brought together to compare. Through comparison of open codes, the team developed a set of focused codes that were iteratively refined through collective discussion. The team then used these focused codes to categorize each sentence from the journals, disputing conflicts through an open discussion until each member of the team was satisfied. This process further refined the focused codes. These discussions resulted in a spreadsheet with each sentence from the journals coded for as many categories (themes) as the group deemed necessary for that section of transcript. Some of these codes are listed in Table 1.

The research team grouped similar codes with each other based on which aspects of the pedagogical process the instructor was discussing. The broad categories included: Teaching, which describes codes in which the instructor is describing what occurred in class; Math, which differentiates instances in which the instructor is explicitly talking about either the students' mathematics (Ms.) or his own (Mi); Reflection, which focused on the successes and failures of implementation toward the desired learning goals; and Tall's Three Worlds, which focused on which of the three worlds the instructor was drawing on in the moment. The codes that we focus on in this section are when the instructor discussed: teaching, focused on the tasks implemented in class (IOLA); teaching, focused on developing specific ideas in the class; teaching, when pedagogical decisions are made; statements about the instructor's mathematics; statements about the students' mathematics; and reflections specifically addressing the students' successes and struggles in developing the intended mathematics.

Table 1. Some focused codes from the iterative coding of the instructor's reflections.

Teaching (T) - Describing what instructor did in class		
	Focus on **tasks**	Tt
	Focus on **developing** ideas	Td
	Responding to student thinking (Formative Assessment)	Tr
	Making **pedagogical** decisions	Tp
Math (M)		
	Instructor	Mi
	Students	Ms
Reflection (R) - Reflecting experiences and on the success/failure of implementation		
	Students	Rs
	Implementation	Ri
	Comparing to Prior Experiences	Rc
Tall's Three Worlds (W) -		
	Embodied	TWe
	Formal	TWf
	Symbolic	TWs

Results and Discussions

After coding the instructor's journals, the team focused on identifying narratives that the codes supported. Through an examination of the codes, we identified several patterns that help explain specific instances of how the instructor made decisions in planning his lessons. One such pattern is diagrammed in Figure 1. In this pattern, statements about developing specific ideas in the classroom (Td) and making pedagogical decisions (Tp) inform the tasks (Tt) in which the instructor engaged the students. The instructor then reflected on the students' activity (Rs), and explicated the insight this allowed him to gain about their mathematics (Ms). Following this, the instructor then drew on his own mathematical understanding (Mi) to make sense of the students' mathematics in the context of his reflection on his own instruction (Ri). This act, in turn, informed the instructor's pedagogical decisions (Tp) and focus on which ideas to develop (Td) as well as a means of developing them through specific task (Tt). Although it is consistent with a few examples from the instructor's journal, this cycle is a generalization of how such a process might unfold and so we expect this process might be different for other instructors. Further, we find that shifts among these codes might provide some insight into how instructors make decisions regarding shifts between Tall's Three Worlds in their instructional decisions.

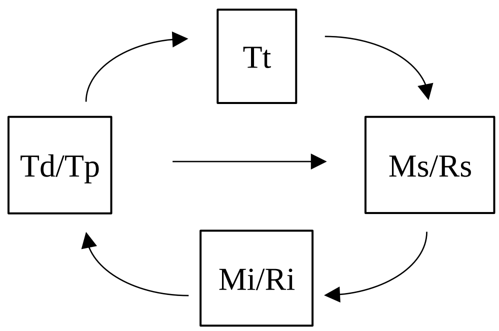

Figure 1. Possible diagram of codes describing instructor's decision-making.

We now provide an example (see Table 2) to show how this pattern of decision-making might unfold. The instructor began this episode by referring to the task that he wanted the students to complete, which comes from the IOLA curriculum intended to support students' development of linear independence, span, and basis (Tt). The instructor then reflected about the students' engagement in the task (Rs) and the mathematical understanding of linear in/dependence and span that they exhibited in their work (Ms). In these lines, he described what he had observed as the students' limitations in completing the table and conjectured why this might be the case, citing limitations of his own instruction in preceding class sessions (Ri). The instructor then responded to this by anticipating an approach that might address what he saw as an issue in his students' understanding. Specifically, he relied on an activity he had developed for himself (Mi) to make sense of the notion of basis. He then concluded that he would implement this task in the next class session (Tt, Td, Tp) and drew on his prior experiences using this task (Mi, Ri). Altogether, this sequence resulted in a shift from focusing on students' generation of examples set in the symbolic world to a focus on the students' embodied notions of linear independence. The instructor saw value in a different way of understanding and anticipated that focusing on this facet of understanding linear in/dependence would support the desired ways of thinking from the students.

Table 2. An excerpt from the instructor's journal.

Excerpt	Tt	Td	Tp	Mi	Ms	Rs	Ri	TWe	TWs

I wanted students to complete U1T4 from IOLA.	1		1						1
Each class finished the task, though some groups had some pretty serious reasoning deficiencies.					1	1			
For instance, very few groups in the first class realized the impossible cases.					1	1			
I think this is a result of rushing through the definition of LI/LD and my failure to support deep geometric thinking about linear dependence.					1		1	1	
I think I can help fix this on Monday by having the students do the "building set" task while focusing on LI/LD.	1	1	1	1					
We've done this task when talking about span and so I think they'll be comfortable with it, I just need to give them time to feel comfortable with thinking about linear dependence spatially.				1		1		1	

Moving between the three worlds

In teaching two sections of linear algebra, David wished to get three fundamental points across. First, he wanted to help students discover that linear independence means there will be infinite solutions to the homogeneous equation. Second, he wanted to demonstrate that linear combinations of linearly independent vectors are unique. Finally, he wanted to help students reason about when and why matrices are invertible by connecting to the importance of basis and forming an appropriate connection between matrices and linear combinations.

David's teaching began smoothly enough, despite some minor complaints about time. Because of this shortage on time, he was unable to introduce the Elementary Row Operations (EROs) that the students had previously asked him about at this time, which disappointed him. He did however, introduce the vector space axioms (formal) and went over a few examples of vector spaces, at which point students finished their prescribed task quickly(Tt). David remarked that the second class seemed much less involved than the first class, perhaps because of the streamlining he did as a result of his experience from the first class (Rs). All groups converted a system of equations to obtain the same solution, though different groups chose different variables and David felt they were unaware of scaling solutions(Mi/Ri). He therefore synthesized the groups' classwork (Td/Tp), helping them focus on free variables (symbolic) and tying their work back to the "getting back home" problem (embodied). He utilized embodied reasoning here, envisioning traveling in a triangle, though perhaps due to rushing through some definitions, the students did not achieve the deep geometric thinking David would have preferred. In attempting to point out that linear combinations of linearly independent vectors are unique, David turned to symbolic reasoning by writing two separate equations for linear combinations of linearly independent vectors(Tt). This achieved the desired effect as students realized that the only way to obtain the zero vector from linearly independent vectors was to make the coefficients all zero, and some students, in David's words, had "really cool ways of thinking about why a=c," (Rs/Ms) and mentions that he wishes he had video of these in-class conversations.

In one particular interview from October 11, David described ways in which he helps his students learn the notion of linear independence. He leveraged the power of the embodied and symbolic worlds as resources to help students understand the formal world more completely. He stated his main goal was to help students understand that matrices are invertible if and only if their column vectors form a basis, that is, if and only if their column vectors are linearly independent (and span the field).

It seems that throughout the interview David utilized embodied reasoning as a resource to help students understand difficult concepts. Early on, he mentioned when students "can't see" some concept he is teaching, he "takes it to a more familiar analog," and furthermore shows pictures of vectors that challenge student intuition. At one point, David talked about helping a student in office hours refine their reasoning about linear independence through embodied use of markers pointing in particular directions. That is, he asked the student if she could get to various places on the desk traveling only in lines parallel to the two markers he had laid out pointing in specific directions. It seems David utilized the embodied world, supported by his example with the markers, to inform his student's understanding about what the more formal terminology of "linear independence" meant.

Earlier in the interview, David stated his wish to help students develop math based on the representations they're already using, as well as their own natural tendencies (orientation). To this end, he asked students in class for their thoughts on a formal idea they're discussing as a class. For example, he wrote "What does it mean to be invertible?" on the board at one point, but rather than having students memorize the formal definition, he asked them for input and wrote the facts they suggested on the board. In this process, he guided them toward the idea of linear transformation which they had already covered, in order to use their previous embodied and symbolic reasoning to inform this new formal idea they were moving towards.

Finally, David suggested he used symbolic work to help reinforce formal ideas. In answer to an interviewer probe, David agreed that he felt that "pictorially" students were fine, but had trouble with representations that "were not just a symbol" but were more strongly "rooted in the definition". In response to this question, David gave an example of another student he asked, "If $\begin{bmatrix}1\\0\end{bmatrix}$ goes to $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ goes to $\begin{bmatrix}1\\1\\2\end{bmatrix}$, where does $\begin{bmatrix}x\\y\end{bmatrix}$ go? The student took an algebraic approach and felt this was intuitive, but still missed the connection to linear dependence. With some help from the instructor, however, sometimes this symbolic reasoning can still inform this formal definition. In doing similar problems, this student answered quickly but her work was disorganized, and therefore confused her. David reorganized the symbolic calculations into a single line with reasoning about linear dependence and transformations informing this symbolic flow of logic. It seemed that David once again utilized symbolic reasoning to help the student cross a bridge from the symbolic world into the beginnings of formal reasoning.

Thus, David builds upon students' prior mathematical knowledge, and utilizes first embodied reasoning to inform the formal definition. Once students are comfortable with embodied reasoning, David begins utilizing tools from the symbolic world to further inform student reasoning, switching back to embodied reasoning (see markers example, or Geometric Sketch Pad (GSP) usage) when students experience further difficulty in reasoning. In class, David successfully led students to symbolically reason about the uniqueness of linear combinations of linearly independent vectors, his second of three main goals.

Not all such teaching was smooth however. One serious reasoning deficiency was that students did not realize some of the cases were impossible, such as writing down three linearly independent vectors in $\mathbb{R}^2$. David first attempted to fix this by helping students think about a matrix times a vector as a linear transformation via a "building set" task they had previously completed while thinking about span rather than linear independence (Tt). He also wanted students to be able to think about the product as a linear combination of column vectors(symbolic), so that students would realize the importance of basis in conjunction with the invertibility of matrices, as well as in conjunction with linear transformations. In particular, column vectors must form a basis for the matrix and the associated transformation to be invertible(formal). However, students had a hard time understanding David's point that a matrix isn't a transformation until you do something with it, and that bases have to be named. He was just trying to explain to them the difference between a simple symbol and the activity that is sparked by that symbol(Ri/Mi)—the difference between symbolic and embodied reasoning. As he felt this point was lost on them, in his second attempt he used GSP (embodied) to actually demonstrate the distinction between symbols and the action the symbol is taken to signify. Asking students to name a vector led them to realize they needed to see axes in order to name a vector(Rs/Ms). David was then able to start with the standard basis axes, then distort them with GSP to a different basis and ask students to come up with a linear combination for the same vector once again. Through embodied work with GSP, students came to understand that with a different basis, a completely different matrix can stand for the same linear transformation, and that a matrix isn't a linear transformation until the bases are established. Thus, GSP in conjunction with David's directed teaching successfully effected a transition in the students from embodied to symbolic reasoning. Showing students (embodied) the difference proved much more effective than just trying to explain the difference(symbolic/formal).

Concluding Remarks

In both accounts, it appears that David roughly followed the cycle laid out in Figure 1, where he utilized developed tasks to draw out or further refine students' existing mathematics. This affords him the opportunity to utilize his own knowledge of mathematics to reflect on his instruction and make inferences about his students' mathematics. Armed with this knowledge, he can then make more appropriate pedagogical decisions and ideas within the classroom. While David notes that students were not typically comfortable shifting between Tall's three Worlds (i.e., formal was most difficult for them while embodied was most accessible, with symbolic somewhere in the middle), in contrast, David himself shifted between worlds regularly in an effort to teach his students. As seen in the October 11 interview, it appeared that David also utilized the embodied world as a resource to help develop the symbolic world, and then used both of these as resources to begin to approach the formal world. Thus, Tall's three worlds functioned as a resource to help David meet his goal of instructing students about the relationship between linear independence and invertibility.

This work provides a foundation for further investigations into linear algebra instructors' decisions, especially those decisions about shifting between representations in each of Tall's three worlds. This research also provides insight into how we might frame instructional decision making more generally – beyond the context of linear algebra instruction. For instance, the decision-making diagram in Figure 1 could be used in any context, though we found it useful in this study to help focus on shifts between Tall's three worlds.

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