Could Algebra be the Root of Problems in Calculus Courses?

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Calculus serves as the gateway for most STEM degrees. Due to students' challenges successfully completing calculus, more than half of students are deterred from a career in STEM. Our preliminary investigation indicates that students' difficulties with algebra cause significant problems in many first-year math courses. The aim of this paper is to investigate in what ways the difficulties with algebra impact students' success in calculus.

Keywords: Algebra, Calculus, common errors, accommodation

Introduction

Calculus occupies the position of gatekeeper to disciplines in STEM since at least one calculus course is required for all STEM majors. "For too many students, this requirement is either an insurmountable obstacle or—more subtly—a great discourager from the pursuit of fields that build upon the insights of mathematics" (Bressoud, Mesa, & Rasmussen, 2015, p. v). Research has shown that negative experiences encountered in gatekeeper or introductory math and science courses are a major factor in the national problem of significant attrition (more than half) of declared STEM majors (Crisp, Nora, & Taggart, 2009; Mervis, 2010). Studies by Stewart & Reeder (2017a; 2017b) suggest that college students' weaknesses with high school algebra play a major role in their success in their first-year math courses.

Although research on students' difficulties with algebra in school has been well documented (e.g. Kieran, 1992; Hoch & Dreyfus, 2004), research on these difficulties and their impact at university level are scarce. Stacey, Chick, and Kendal (2004) discussed the main problems of algebra in school algebra, little was mentioned in the way of consequences for college level mathematics. Research has catalogued common errors in computation and algebra (Ashlock, 2010; Booth, Barbieri, Eyer, & Pare-Blagoev, 2014; De Morgan, 1910). Our findings parallel these categorizations and document that these errors continue to persist in college level mathematics work, potentially complicating student success in college mathematics courses (Stewart & Reeder, 2017b). As Author (2017, p. vii) points out: "Many college instructors are facing this dilemma every day. Students who seemingly follow more complex mathematical concepts, are unable to proceed as problems, for example involving fractions, will soon let them down." We suggest that challenges students have with the high school algebra content that is embedded in calculus problems are a major cause of failure for many Calculus students.

The goal of our research is to understand how students' difficulties with algebra impact their work in calculus problems. For this study, Calculus students were given algebra tasks and calculus tasks with algebra embedded to help answer the following research questions: (a) What were the most common algebra problems in both the algebra and calculus tasks? (b) What were the students' perceptions of their challenges with algebra and calculus related to these tasks?

Theoretical framework

Piaget's (1952) theory of accommodation and assimilation as a theoretical framework was employed for this study. A schema (mental structure) serves two purposes: "It integrates existing knowledge, and it is a tool for acquisition of new knowledge" (Skemp, 1971, p. 39). When new situations and experiences are encountered, the human brain deals with it by either accommodation or assimilation; the structure of the schema must change to adapt to the new situation, “this may be difficult; and if it fails, the new experience can no longer be successfully interpreted, and adaptive behavior breaks down- the individual cannot cope” (p. 44). In this way, how we understand concepts is constantly changing and adapting as we are presented with new information, experience things, and learn new concepts. While assimilation is easier and often produces a feeling of mastery, accommodation is difficult. Vinner (1988) stated that “very often (and specially in mathematics) the cognitive structure of the learner is not suitable for incorporating the new material” (p. 594). He believed that acquisition of new mathematical concepts in more advanced settings requires accommodation, since “a concept which seems quite simple to the mathematician can be difficult for the student to accommodate” (p. 606). He further believed that the lack of attention to accommodation will lead into situations where “certain concepts are not conceived by the students the way we expected” (p. 593). Skemp (1979) introduced two further notions: *expansion* and *reconstruction*. He clarified that “our schemas grow by expanding existing concepts and by forming new ones” (p. 126). Sometimes, however, we may encounter a situation for which we have a relevant schema which is not adequate. If we are unable to avoid such situations, we need to re-construct our schema. This is “disruptive, unwelcome, and difficult: because while this is going on, we are unable to use our schemas effectively for directing our actions” (p. 126). We suggest that success in calculus requires expanding and reconstructing schemas about algebra in order to make sense of the calculus contexts in which they appear.

Method

This qualitative research study involved 275 Calculus I students at a university in the Southwest US at the end of their 16-week course. Students were asked to solve three common Calculus I tasks and four algebra tasks, identify what caused them the most challenge, algebra or calculus, and provide a brief discussion about what challenged them while solving the tasks (see Table 1). The algebra tasks were designed such that they focused on the algebra students would encounter while solving the calculus tasks. Students were asked to solve the calculus tasks first (30 min) and were given the algebra only after their calculus problems were completed (20 min). The open response question was provided last.

Once all data were collected, it was de-identified and incomplete data sets were removed. The result was N = 84 complete sets of data. The research team (four individuals) met to analyze each problem to develop an initial codebook. The initial codebook was used by researchers to code ten sets of data independently for both calculus mistakes in the calculus problems and algebra mistakes in both the calculus and algebra problems. A second meeting of the research team focused on establishing the code book and inter-coder reliability. With an established codebook (see Table 2), each set of problems were analyzed and coded independently by two members of the research team. Each team met to review the codes and establish 100% agreement.

Table 1. The Calculus and Algebra tasks.

Calculus tasks	Algebra tasks
1. Implicitly differentiate. $\sqrt{xy} = 1 + x^2y$ 2. Find the critical numbers of the function $f(t) = t\sqrt{4 - t^2}$ 3. Evaluate the limit.	1. Solve for y. $5 + xy = 10 + x^2y$ 2. Solve for y. $\frac{1}{2\sqrt{5x}}(5 + xy) = 10x + x^2y$ 3. Solve for t.

$$\lim_{t\to 0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t}$$	$$\frac{1}{\sqrt{t+1}}-\frac{1}{t}=0$$ 4. Solve for y. $$\frac{2y^2}{2\sqrt{y^2-9}}+\sqrt{y^2-9}=0$$
My main problem with the test was: Algebra ☐ Calculus ☐	
Please write a comment relevant to your experience in taking this test.	

Table 2. Potential errors for Calculus and Algebra contexts.

Possible Calculus Errors	Possible Algebra Errors	Other Possible Error
1. Power Rule 2. Product/Quotient rule 3. Chain Rule 4. Process of Implicit 5. Interpret Critical Numbers (set =0) 6. Undefined points are Critical 7. Taking the limit	8. Convert radical to exponent 9. Exponent Operations 10. Balance Points 11. Distributive Property 12. Combining Like Terms 13. Cancelling 14. Factoring 15. Simplifying nested fractions 16. Sign error 17. Operations with radicals 18. Finding Common Denominators 19. Recognizing undefined values 20. Conjugating Rational Fractions 21. Quadratic Functions 22. Operations with Fractions	23. Blank 24. Incomplete Calculus 25. Incomplete Algebra 26. Computation 27. Avoiding Algebra 28. Avoid Calculus 29. Miscellaneous 30. Isolating Variables

Results

Our first research question focused on determining the most common errors students made while completing the algebra tasks and calculus tasks while our second question focused on the students' perceptions of their challenges with algebra and calculus. As such, the results are presented in two sections: Research Question 1 and Research Question 2.

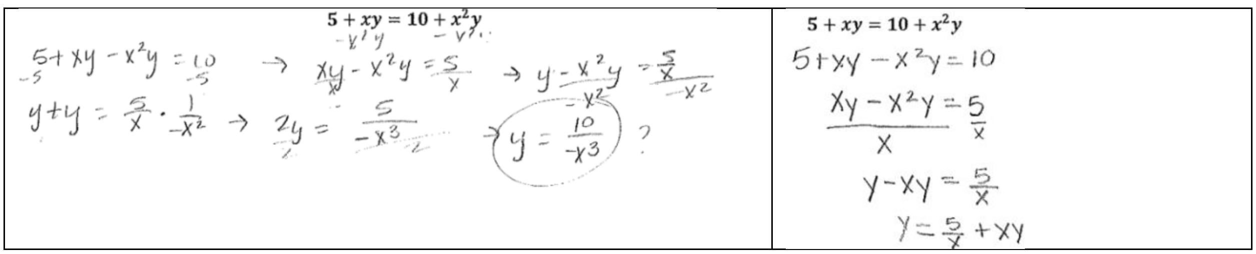
Research Question 1

Analysis of the algebra and calculus tasks revealed that the student errors were numerous and significant with algebra in both sets of tasks and calculus related errors were frequent in the calculus tasks as well. The most common algebra errors made in both sets of tasks were problems working across the balance point in equations, cancelling, operations with radicals, appropriate application of the distributive property, and incomplete algebra (work that was not completed due to confusion). While the students' work with the calculus tasks were replete with algebra errors, they also made many calculus errors. The most common among these were correctly taking the derivative implicitly, using the product rule properly, failing to identify undefined points as critical, incorrectly taking the limit, and avoiding algebra.

Analysis of Algebra tasks

The first algebra task directed participants to solve for y. This problem required collecting like terms and then factoring to isolate the variable *y*. The most common mistakes illustrated that participants had an incomplete conceptual understanding of what it meant to solve an equation, either because they did not isolate the *y* variable or because they did not recognize factoring as a strategy that could help isolate the variable. Figure 1 illustrates two examples of these types of mistakes by different students.

The initial mistake by the first student occurred when s/he attempted to divide each side of the equation by - x^2 (see Figure 1(a)). Clearly, the student was attempting to rewrite the left side of the equation in a form which would allow the terms containing *y* to be combined; based on the incorrect work, the student combined these terms. The mistake was failing to recognize that factoring would accomplish this goal while performing operations on both sides of the equation would not. The second student compounded the errors as s/he tried to find a way to combine the two *y* terms. In other words, the student either failed to recognize that manipulating terms was no longer a viable option, or was unable to determine another viable strategy for solving equations. Likewise, the student whose work is presented in Figure 1(b) reached the point where s/he should have shifted strategies from manipulating both sides of the equation to factoring the left side of the equation, but continued to manipulate both sides of the equation instead, which resulted in an equation that was not solved for *y*.

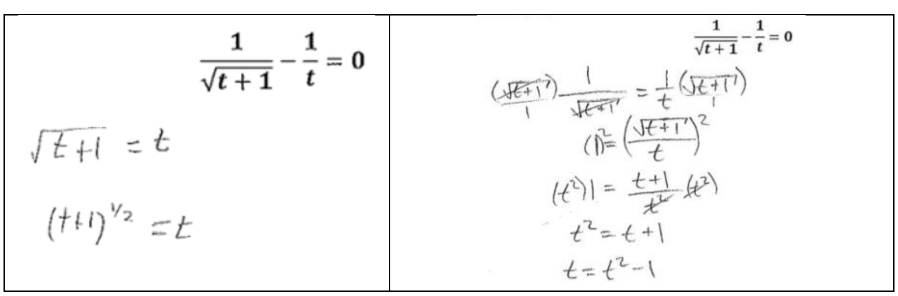


(a) (b)

Figure 1: (a) Student did not recognize factoring as a strategy for solving equations, (b) Student did not isolate y.

In the second algebra task, students had similar issues determining what strategies to use and when to move between strategies to solve the equation. Even when students successfully solved the problem, it sometimes appeared as if strategies were chosen at random and students seemingly solved the equation through determination and perseverance. Because of the multitude of technically correct, but unhelpful strategies that can be employed for task two, more mistakes were made with this task than any of the other algebra tasks.

The difficulty students faced in the third algebra task centered around points at which they needed to change strategies. Determining a strategy that allowed the equation to be rewritten without a radical and a strategy to use to solve the resulting quadratic equation challenged many students (see Figure 2). Interestingly, students were much more likely to simply stop working on task three when they reached one of these decision points than they were to stop working on task one or two.



(a) (b)

Figure 2. (a) Student was not able to work with the radical, (b) Student was unable to determine a strategy for solving the quadratic equation.

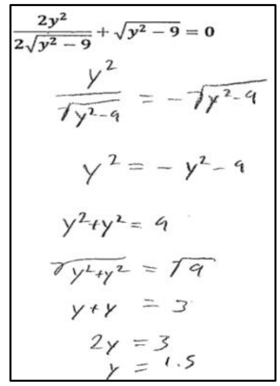
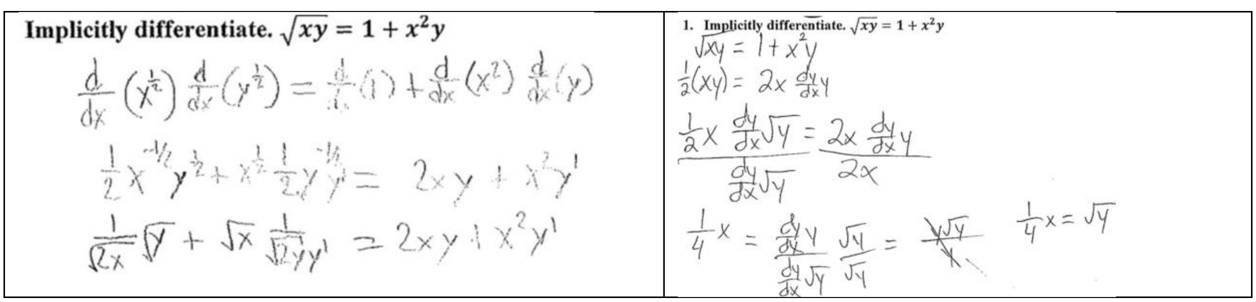


Figure 3. Difficulties with radicals.

Operations with radicals proved to be a major difficulty for students in problem four. Not only did students have difficulty determining how to eliminate the radical, but students were also more likely to make mistakes in earlier algebraic concepts when wrestling with them in conjunction with radical. For example, one student (see Figure 3) properly eliminated the radical through multiplication, but failed to distribute the negative through the resulting binomial; a mistake fortuitously corrected by his/her next mistake. In addition, the student failed to recognize that the two y^2 terms could be combined, and incorrectly assumed that a radical could be, for lack of a better term, distributed to each term within it. It is important to note that many students who correctly applied the distributive property and correctly combined like terms in earlier problems routinely misapplied these procedures in task number four when radicals were involved. As students learn mathematics they build new schema, assimilate and accommodate new information, and expand and reconstruct existing schema. If these schemas are formed around misconceptions or incomplete understandings of mathematical concepts then as they are expanded and reconstructed through the ongoing process of accommodation, students will have continual problems in mathematics.

Analysis of Calculus tasks

In the first calculus task, the most common errors students made were correctly taking the derivative implicitly, using the product rule properly, and failing to complete the necessary algebra correctly. For example, one student (see Figure 4 (a)) incorrectly differentiated each variable separately on both sides of the equation in the first line, and then did not finish solving for y'. Note, that despite the incorrect notation of the first line, the second line appears to contain the correct derivatives.



(a) (b)

Figure 4. (a) Incomplete Algebra (b) Incorrect Chain Rule and Product Rule with disappearing derivative.

In contrast, another student (see Figure 4(b)) did not apply the chain rule properly on the left side or the product rule properly on the right side. Also note the strange algebra in the second and third lines lead to the $\frac{dy}{dx}$ term disappearing, making it impossible to solve for y' as required, so this again is incomplete algebra. Four students total were completely correct for Question 1 as

61 of the 84 students either could not differentiate implicitly did not correctly apply the product rule, or did not solve for $\frac{dy}{dx}$.

In the second calculus task, the typical errors were failing to identify undefined points as critical, incorrectly applying the product rule or failing to apply it altogether, and failing to complete the calculus portion of the task. For example, one student could not complete the calculus due to difficulty with the product rule (see Figure 5). However, it is notable that s/he successfully identified the mistake and gave reasonable instructions for how the problem *should* be solved. Note s/he finds two of the critical points (± 2), but this is somewhat accidental, as s/he finds these by setting his incorrect derivative to 0, when the points ± 2 should be obtained from finding the points at which the derivative is undefined. The points obtained from setting the correct derivative to 0 should be $\pm\sqrt{2}$. Only two students correctly solved task two.

The two most common errors in the third calculus task were incorrectly taking the limit and avoiding algebra (i.e., actively avoiding rationalizing the numerator). One student (see Figure 6 (a)) used the quotient rule in an inappropriate scenario (perhaps conflating with L'Hopital's Rule) to simplify the limit. S/he followed this very well executed quotient rule with an improper cancellation of one of the *t*'s, which led her/him to assume that the limit does not exist, despite still having *t*'s in both numerator and denominator. A few students (see Figure 6 (b)), incorrectly utilized a limit law to separate the two terms to separate the limit. While this strategy works well if both resulting limits converge, it does not here because the two separate limits both diverge. Note that while this student did not claim that the limit does not exist, s/he appears to have stalled out and never attempted to evaluate the limit. This task resulted in the more correct work from students (7 of 84) but provided the most variation in the types of errors students made.

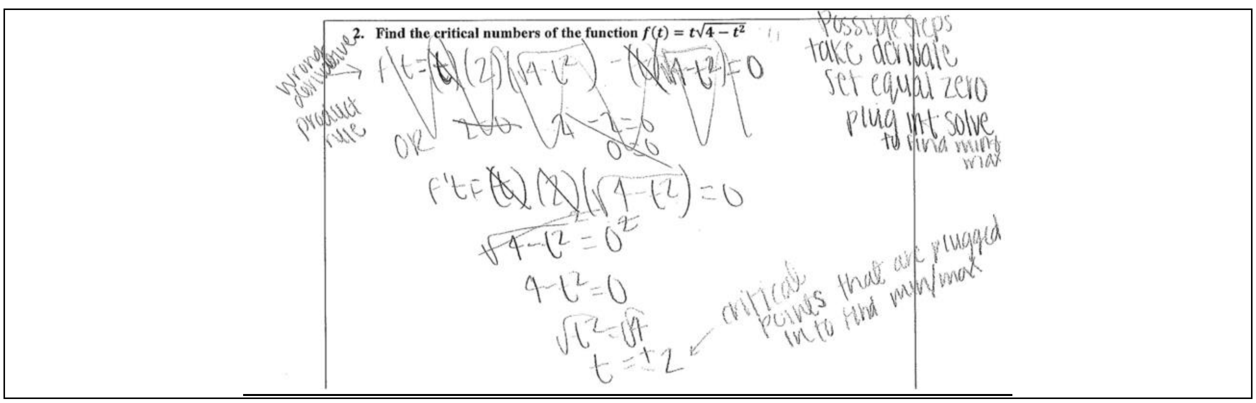
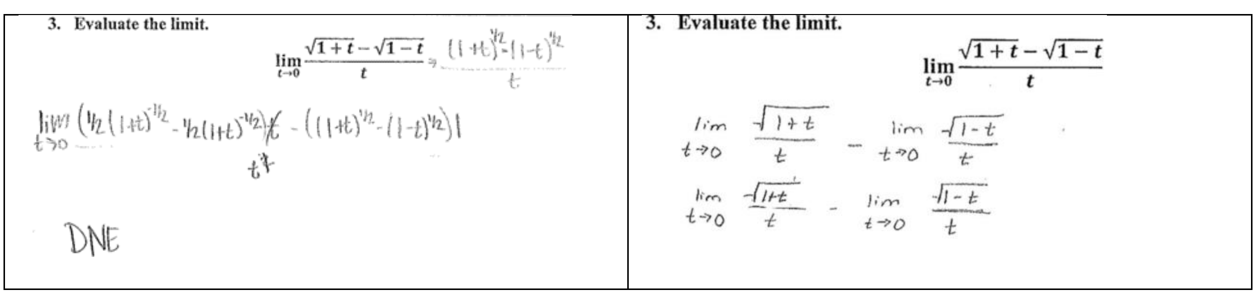


Figure 5. Instructions for how problem should be solved.



(a) (b)

Figure 6. (a) Out-of-context Quotient Rule (b) Incorrect limit law.

As students encounter new concepts in calculus they are no doubt building new schema to accommodate for the news ideas and new mathematics. However, in the midst of dealing with new ideas they must also rely on schemas they developed for algebraic manipulations in the setting of the new concepts they are learning in calculus. If the schema for their algebra understanding are incomplete, then they may present significant challenges for the students as they rely on them to develop understandings of new concepts.

Research Question 2

Analysis of Students' Comments

Our second research question aimed to provide insight on the students' perceptions about their abilities with algebra and calculus as presented in the tasks they were asked to solve. During the one-hour data collection session, students solved three calculus tasks and four algebra tasks and while pressed for time, 73 of the 84 chose to provide a response to our short-answer item. When asked to simply select which gave them more challenge, algebra, calculus, or both, 57% indicated algebra, 31% indicated calculus, while 12% indicated both. Their comments overwhelmingly expressed recognition that algebra causes them difficulties, frustration, anxiety, and in some cases, hopelessness about their abilities to succeed in mathematics. An excerpt of student comments below capture this well:

- *Square roots and fractions can make algebra difficult and confusing. Calculus can be difficult too but there are more steps either before or after the calculus that involve algebra and that can either "make or break" the problem and solution.*
- *I've had a very weak base in Algebra, ultimately leading to a dysfunction in Calculus.*
- *I struggled the most on the algebra portion of the test. However, I struggled with both portions of the test. I felt as if I hadn't learned anything or retained anything in my course of math. I want to be better at math, but I don't know how.*

Concluding Remarks

We hypothesized that students would solve algebra problems largely correctly when these problems were in isolation from calculus, but make predominantly algebraic mistakes in the context of calculus problems with algebra problems embedded. However, we found that our sample of students had difficulty in all aspects of both the algebra and calculus tasks. Students routinely struggled with the isolated algebra tasks as well as the calculus tasks. While the work with the tasks presented students challenges with both calculus and algebra the student responses overwhelmingly indicated they had frustration and concerns with their algebra abilities. In the words of one student "I knew how to start the problem, but couldn't finish because of the difficulty of the algebra involved." This presents a challenge for those of us teaching undergraduate mathematics. Our students may have the prerequisite knowledge, but it may not be strong enough to function as a versatile tool in calculus as expected or required. Certainly, further research is needed to examine students' abilities with algebra and its' impact on their success in undergraduate mathematics. We are in the process of designing further studies by interviewing students and mathematics professors in order to gain a better appreciation of students' difficulties. Ultimately, we would like to create a model of intervention to remedy calculus students' struggles with algebra.

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