Schema Development in an Introductory Topology Proof

Ashley Berger
University of Oklahoma

Sepideh Stewart
University of Oklahoma

This is an exploratory study into schema development of introductory topology students. We discuss Skemp (1987) and Dubinsky and McDonald's (2001) definitions of schema and how they fit with Piaget and Garcia's (1989) triad framework. We employed these theoretical instances on the idea of schema to analyze students' responses to a final exam problem about a basis for the product topology on a product space. Our analysis indicates that the majority of the students were still in the beginning stages of schema development by the end of the semester in a topology course.

Keywords: schema, Topology, basis

Theoretical background

Advanced mathematics courses are often difficult for undergraduate students to transition into and research on student difficulties on advanced courses, especially on topology, are scarce. The overarching goal of this project is to build a theoretical framework investigating the differences between expert mathematicians and novice undergraduate students' schemas in topology. We also would like to be able to investigate how students' schemas develop (Piaget's accommodation) and how interactions with peers and instructors affect that development. In this case study, we embark on this journey by examining undergraduate students' proof attempts involving a basis for the product topology on $X \times Y$.

We will employ the idea of schema to gain more insight into the transition towards advanced mathematics, specifically towards topology. Although there are multiple definitions of schema currently in the literature, in this study we will mostly focus on Skemp's version. In 1962, Skemp argued for the need of a valid learning theory that was developed in classrooms:

> A theory is required which takes account (among other things) of the systematic development of an organised body of knowledge, which not only integrates what has been learnt, but is a major factor in new learning: as when a knowledge of arithmetic makes possible the learning of algebra, and when this knowledge of algebra is subsequently used for the understanding of analytical geometry. (p. 133)

Skemp (1962, p. 133) defines schema as the "organised body of knowledge" that integrates existing knowledge and is a major factor for new learning. Additionally, he defines and compares schematic learning to rote learning (non-schematic learning). Unsurprisingly, he finds that "Schematic learning has a triple effect: more efficient current learning, preparation for future learning, and automatic revision of past learning." (p. 140)

Skemp (1987) gives a more detailed definition of schema in his chapter, "The Idea of a Schema". He describes a system where concepts are embedded in a hierarchical structure of other concepts, where levels in the structure are classifications of concepts. For example, a train can be classified as a mode of transportation and can contribute to one's concept of transportation. We can also pair concepts together, giving a relation between them, which we can also classify. Additionally, we can look at transformations of concepts, which can be combined to make other transformations. What makes this hierarchical structure of concepts, relations, and transformations so deep and complex is the fact that these classifications are not unique, giving way to multiple hierarchical structures, which can be interrelated. When components of these conceptual structures come together to make a structure that would not be realized by only looking at the individual components, we call this resulting structure a schema. Skemp (1987) claims that a schema integrates existing knowledge, serves as a tool for future learning, and makes understanding possible. Without a suitable schema, students will have difficulty in understanding or making sense of new concepts. Skemp (1987) used topology in his work for the reason that "the relevant schema can be quickly built up, whereas most mathematical ones take longer." (p. 30) Although this study focuses on a more advanced topology question than Skemp did, we still believe that topology offers ideal topics to observe schema development with since most students do not encounter topology until late in their undergraduate work.

Another definition of schema is embedded in APOS Theory (Dubinsky & McDonald, 2001). Actions, processes, and objects are used to define a schema. Actions are external transformations of objects that become processes once internalized. After an individual becomes aware of a process and the transformations that can act on it, the process has become an object itself. Dubinsky and McDonald (2001) continue on to define schema:

> Finally, a schema for a certain mathematical concept is an individual's collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept. This framework must be coherent in the sense that it gives, explicitly or implicitly, means of determining which phenomena are in the scope of the schema and which are not. (p. 3)

Clark et al. (1997) discussed an application of Piaget and Garcia's (1989) triad framework, Intra, Inter, and Trans, to the chain rule in Calculus. This triad is a theory for schema development within the context of APOS. Before a schema is coherent, it must go through these three stages. In the Intra stage, an object is thought of in isolation from other actions, processes, or objects. Once relationships are seen between the object and other actions, processes, objects, and schemas, the individual is in the Inter stage, also known as a pre-schema. In the Trans stage, a coherent structure begins to underlie the relationships from the Inter stage, and there now exists a schema for the original object in question.

As an example, consider the development of a schema for a topology. Working purely within the definition of a topology and considering basic examples is in the Intra stage. The schema enters the Inter stage once connections between the definition and previous knowledge are made. This includes more complex examples and possibly basic proofs. Viewing a topology as how open sets are defined for a topological space and being able to apply that in more complicated proofs demonstrates ideas in the Trans stage. This triad will be used as a place to begin analyzing schema development for a proof in an introductory topology course.

We view Piaget and Garcia's (1989) triad framework as a continuous spectrum for developing a schema. Dubinsky and McDonald's (2001) definition of schema overlaps with only the Trans stage since that is when a coherent structure appears. In comparison, Skemp's (1987) definition of schema not only overlaps with the Trans stage, but all stages of the triad framework. In our view, an idea does not have to be fully developed or correct in order to be a part of a schema. Our research question for this project is "With respect to the triad spectrum, how developed are introductory topology students' schemas for a basis for a topology?"

Method

This is a case study into introductory topology students' thinking about a basis for a topology. Eleven final exams were collected and de-identified from a senior-level undergraduate topology class at a research university in the Southwest US. This study focuses on the first of the nine exam questions, shown in Figure 1.

1. (a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. Define the product topology $\mathcal{T}$ on $X \times Y$.

   (b) Show that the projection map $p_X : X \times Y \to X$ defined by $p_X(x, y) = x$ is an open map.

Figure 1. Question 1. Define and use the product topology on a product space.

We chose this question for a couple of different reasons. First, it is structured such that students who are in-between the Intra and Inter stages of their schema development for a topology generated by a basis can still answer part a. Then part b requires students to be at least in the Inter stage of schema development. This question quickly reveals students whose schemas are still in the Intra stage.

Compared to other questions on the exam, this problem is more consistent with content from a typical introductory topology class. It would be unusual if the product topology on $X \times Y$ and the use of a basis did not appear in a beginning topology course, and therefore this problem is one that can be considered for use in future expansions of this study. This problem was also the first on the exam and therefore all of the students made an attempt on it.

The data was initially coded by identifying the types of errors made in each part of the problem (see Table 1). We then went through a second round of coding for consistency and grouped the responses together based on these errors and attempted to analyze them with the triad spectrum.

Table 1. Types of errors.

Code	Description	Percentage of Students with Error
B	Left blank or contributed no original thoughts	9.1%
IN	Issues with notation	36.4%
IL	Issue of beginning proof with conclusion/other incorrect logical statement	45.5%
NB	No reference to a basis	63.6%
LC	Lacking clarity	72.7%
LL	Lacking logical flow	18.2%
LD	Lacking direction	9.1%

Results and Discussion

The product topology on $X \times Y$ can be defined using the collection $\beta = \{U \times V | U \in \mathrm{T}_X, V \in \mathrm{T}_Y\}$ as a basis. The proof for part b involves three main components:

A. Noting that all open sets can be written as a union of basis elements (this part may be considered part of the definition of a basis depending on how it was presented in class)
B. Noting that the projection of a union is a union of projections
C. Showing the projection map is an open map for basis elements

We understand it is up to each instructor as to how detailed students' proofs should be, but these three components should at least be noted somehow in the proof. Figure 2 gives an overview of the proof schema. The arrows in the figure indicate previous knowledge that is needed in order to complete parts of the problem.

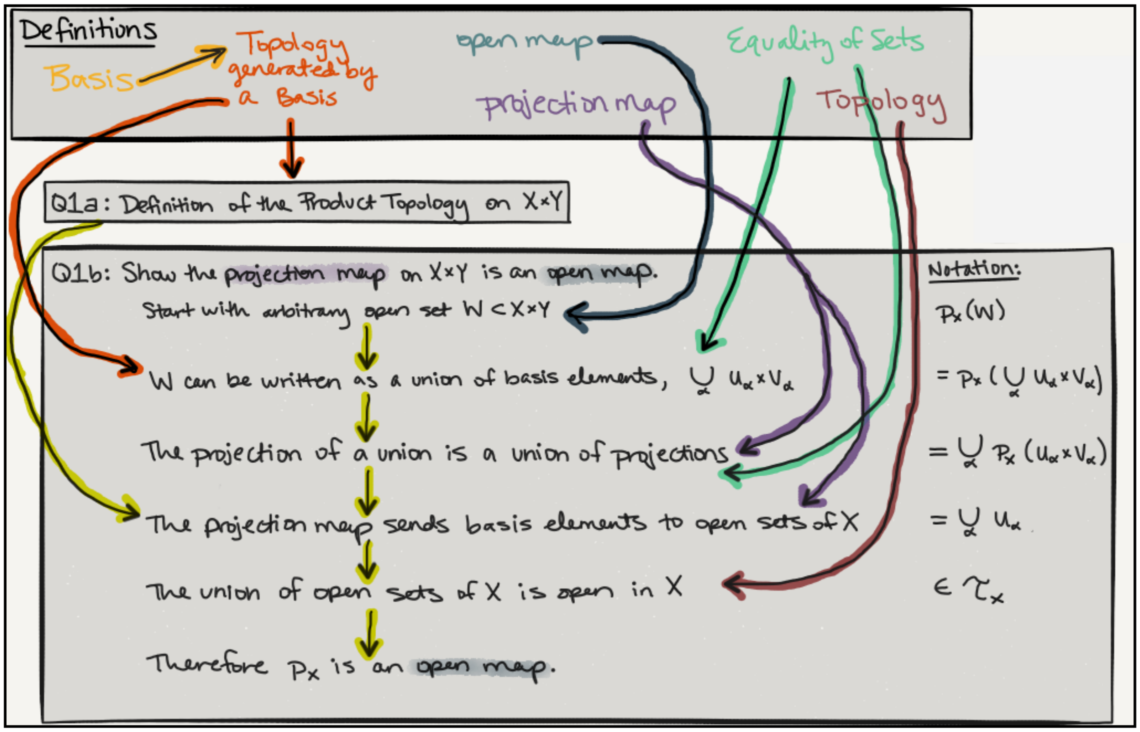
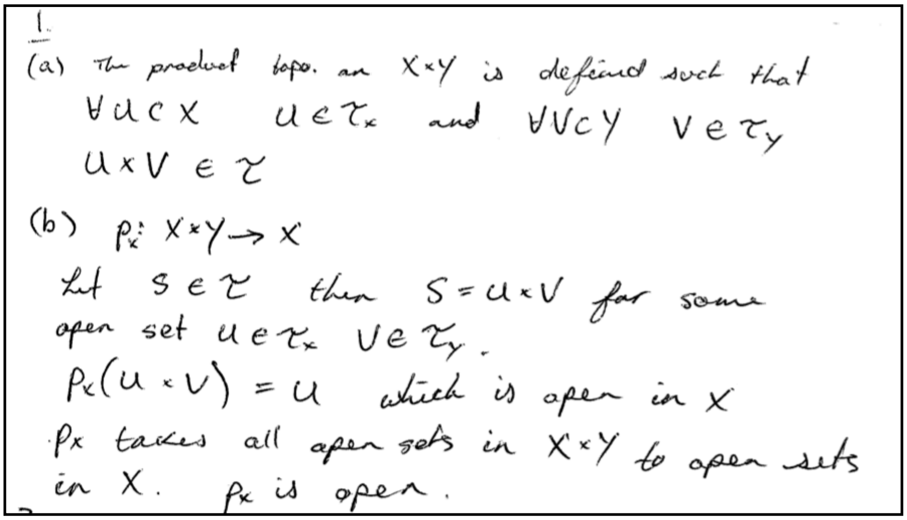


Figure 2. A proof schema for the problem.

Seven of the eleven students did not use a basis to define the product topology on $X \times Y$ and six of those seven students claimed that the topology on $X \times Y$ is $\mathrm{T}_{X \times Y} = \{U \times V | U \in \mathrm{T}_X, V \in \mathrm{T}_Y\}$. A typical response of this type is shown in Figure 3.

The following argument demonstrates why this response cannot be the topology on $X \times Y$ and why a basis is needed. Let $\beta = \{U \times V | U \in \mathrm{T}_X, V \in \mathrm{T}_Y\}$ be the basis for $\mathrm{T}_{X \times Y}$. $U_1 \times V_1$ and $U_2 \times V_2$ are both elements of β and therefore are also elements of $\mathrm{T}_{X \times Y}$. By the definition of a topology, $(U_1 \times V_1) \cup (U_2 \times V_2)$ is also an element of $\mathrm{T}_{X \times Y}$. Note, however, that the union is not of the same form as elements of β and cannot be in β, as shown in Figure 4. So β cannot be the entire topology on $X \times Y$.

Since the proof for part b depends on the use of a basis, the seven students who did not use a basis in part a were unable to write a complete proof for part b. They often showed component C of the proof but did not include components A or B. The students who had this type of response may not see the need for a basis, when it is appropriate to use one, or how to make use of it. There is a disconnect between this problem and the definition of a topology generated by a basis. Therefore these students' basis schemas are, at best, in the Intra stage of schema development. They have not reached the Inter stage since they are unable to connect a basis with other knowledge.



1.

(a) The product topo. on $X \times Y$ is defined such that
$\forall U \subset X \quad U \in \tau_x$ and $\forall V \subset Y \quad V \in \tau_y$
$U \times V \in \tau$

(b) $p_x : X \times Y \to X$
Let $S \in \tau$ then $S = U \times V$ for some open set $U \in \tau_x$ $V \in \tau_y$.
$p_x(U \times V) = U$ which is open in X
p_x takes all open sets in $X \times Y$ to open sets in X. p_x is open.

Figure 3. A response that is not past the Intra stage.

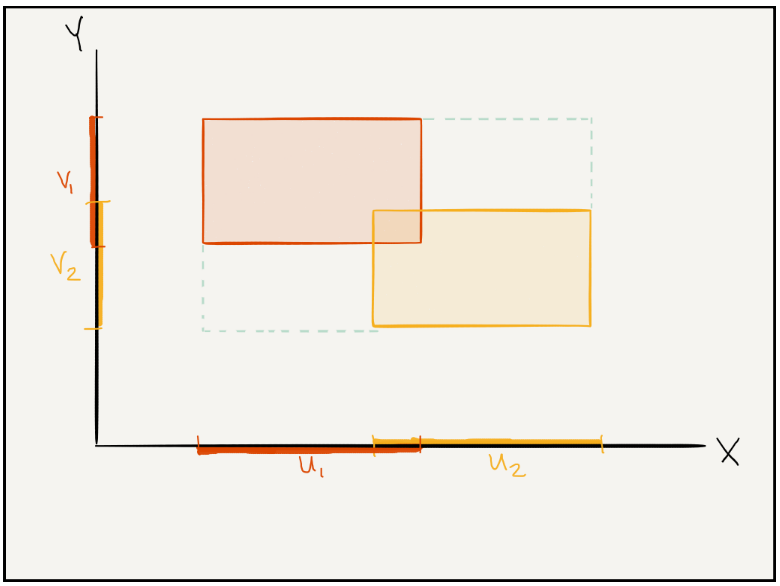


Figure 4. A visual representation of the need for a basis.

The four students who did make use of the basis had problems with incomplete proofs and notation. They would write the proof for basis elements only and then immediately jump to the conclusion of the proof without addressing components A or B of the proof. Such an example is in Figure 5. Whether or not the proof is considered to be correct depends on the instructor and the classroom norms. For this study, however, we are not as concerned about the validity of the proof as much as what it does (or in this case, does not) tell us about the student's schema of a basis. The use of the word "basis" can be used as a substitute for component A of the proof, but we cannot assume that the student did or did not understand this. The same goes for component B, which may or may not have been considered trivial in the class. We can say that this student has reached the Inter stage of basis schema development since they could relate a basis with other actions, processes, and objects, but due to the minimal amount of details in their proof, we cannot make any conclusions past this stage about their level of understanding.

1b: Take $P_x : X \times Y \to X$ $\quad P_x(x,y) = x$.

By construction, $\mathcal{B}$ is a basis for $\mathcal{T}_{X \times Y}$. Take any $B \in \mathcal{B}$

$B = U \times V$ for some $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$

$P_x(B) = P_x(U \times V) = U \in \mathcal{T}_X$ by assumption.

Thus, P_x takes the basis to open sets, so

$P_x : X \times Y \to X$ is an open map.

Figure 5. A response that has reached the Inter stage.

There were four students who had notational issues and nearly all students could have made their arguments more clear. An interesting example of this is in Figure 6. The student in this example incorrectly used $A{\times}B$ as their arbitrary open set of $X{\times}Y$, yet still included component A of the proof by saying that $A{\times}B$ is a union of basis elements. This indicates that the student had an understanding of the need for component A in their proof schema, but they did not understand how to denote the arbitrary open set. The student has a coherent proof structure here, but their argument could be improved with some corrections in notation. This student's response shows that they have reached the Trans stage, but there are still some notational gaps to fill in in their overall schema.

1) a) The product topology Υ is the topology induced by the basis $\mathcal{B} = \{ U \times V \mid U \in \Upsilon_X \text{ and } V \in \Upsilon_Y \}$

b) Let $A \times B$ be an open set in $X \times Y$

$P_x(A \times B) = A$

By the product topology, $A \times B = \bigcup U_x \times U_y$, where U_x and U_y are open sets in X and Y, respectively, and form a basis. $A \times B = \bigcup_{\alpha \in A} (U_x^\alpha \times U_y^\alpha) \to A = \bigcup_{\alpha \in A} U_x^\alpha$.

Since $P_x(A \times B) = \bigcup_{\alpha \in A} U_x^\alpha$, which is open in X, P_x maps any open set in $X \times X$ to an open set in X, which makes P_x an open map

Figure 6. A response that has reached the Trans stage.

Concluding Remarks

The three examples discussed in this study demonstrate three different places along the triad spectrum where student's schemas could be. Even though this problem came from a final exam at the end of the semester, a majority of the students surprisingly were still `at the Intra stage or lower in their schema development for a basis. We cannot comment on why this is since we did not collect any data regarding the norms of the class that these participants were in. This also means that we cannot know what was considered to be trivial in the course, making it difficult to analyze student's responses that are similar to Figure 5. These schemas may or may not include the components that were replaced with equivalent, but highly simplified, statements. We also do not know how much the instructor emphasized the need for a basis for certain topologies or whether or not the students had seen this problem on a previous homework assignment, both of which would affect the students' schemas.

The other limitation to this study is that it is impossible to physically see the schema of another person, so at best we can only make conjectures about participants' schema development, especially since we analyzed written proofs. Interactions with participants will be more informative in future work.

The next steps for expanding our project include interactions with the participants, data collection that occurs at the beginning and the end of a semester, and interactions between participants in either a partner or group setting. We hope to have participants explain their schemas out loud to us or a peer and to observe progress in the development of their schemas over time. We also will be asking a wider variety of questions over introductory topics to gain a better sense of which topics are more challenging for undergraduate students.

References

Clark, J. M., Cordero, F., Cottrill, J., Czarnocha, B., DeVries, D. J., John, D. S., Tolias, G. & Vidakovic, D. (1997). Constructing a schema: The case of the chain rule? *The Journal of Mathematical Behavior*, 16(4), 345-364.

Dubinsky, E., & McDonald, M. A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In D. Holton (Ed.), *The teaching and learning of mathematics at university level* (275-282). Springer Netherlands.

Piaget, J. (1950). *The psychology of intelligence* (English edition). London : Routledge and Kegan Paul.

Piaget, J., & Garcia, R. (1989). *Psychogenesis and the history of science*. Columbia University Press.

Skemp, R. R. (1962). The need for a schematic learning theory. *British Journal of Educational Psychology*, 32(P2), 133-142.

Skemp, R. R. (1987). *The psychology of learning mathematics* (Expanded American ed.). Hillsdale, N.J.: L. Erlbaum Associates.