Generalizing in Combinatorics Through Categorization

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In this report we discuss students generalizing within a combinatorial setting. To facilitate reflection on prior activity, we prompted students in a teaching and a design experiment to categorize a myriad of problems they had previously engaged in. We will discuss the combinatorial underpinnings behind the students' generalizations according to Lockwood's (2013) model for combinatorial understanding. We saw that the students were able to produce generalizations of various basic combinatorial problems while each maintaining different understandings of the combinatorial structures. We conclude by discussing uniformity in the students' reasoning pertaining to combinations and the productive nature of such discussions.

Key words: generalization, combinatorics, combinations, permutations

Introduction

The activity of generalization is integral to mathematical thought, reaching all education levels (Amit & Klass-Tsirulnikov, 2005; Lannin, 2005; Peirce, 1902). While there is a growing body of literature on student generalization, we still have much to learn about fostering productive generalizing activity in various contexts. Through a multi-phase study, we sought to better understand students' generalizing activity in a combinatorial setting. Combinatorics provides a natural setting for generalization, as counting problems are often accessible yet challenging (Kapur, 1970; Tucker, 2002). These accessible problems provide a natural structure from which students may generalize. In this report, we discuss the results of student engagement in a categorization task designed to facilitate reflection on prior work with various counting problems. The students collectively produced sophisticated generalizations while individually maintaining unique combinatorial understandings. We discuss the various nuances of their understandings as well as some affordances of attending to certain combinatorial structures.

We will discuss the students' generalizing activity in accordance with Lockwood's (2013) model for combinatorial thought. Such an analysis provides a deeper understanding of the potential source material for students' generalizations in combinatorics. We seek to answer the following research question: What do students attend to combinatorially as they generalize?

Literature Review

Generalization

Generalization has been recognized as a key aspect of mathematical activity by both researchers (Amit & Klass-Tsirulnikov, 2005; Davydov, 1990; Ellis, 2007b; Vygotsky, 1986) and policymakers (Council of Chief state School Officers, 2010). While much of the literature on student generalization focuses on algebraic contexts (Amit & Neria, 2008; Becker & Rivera, 2006; Carpenter, Franke & Levi, 2003; Ellis, 2007a/2007b; Radford, 2006/2008; Rivera, 2010; Rivera & Becker, 2007/2008), more recent studies have looked at undergraduate student generalizations in calculus (Dorko, 2016; Dorko & Lockwood, 2016; Dorko & Weber, 2014; Fisher, 2007; Jones and Dorko, 2015; Kabael, 2011) and combinatorics (Lockwood & Reed, 2016). Lockwood and Reed (2016) first investigated generalization in combinatorics by demonstrating two students that produced similar generalizations while holding vastly different meanings for their constructs. This report contributes to the growing body of literature by providing instances of generalization being rooted in various nuanced combinatorial understandings.

Combinatorial Reasoning

Though combinatorics provides accessible and deep tasks (Kapur, 1970; Tucker, 2002), students struggle reasoning combinatorially (Batanero, Navarro-Pelayo, & Godino, 1997; Eizenberg & Zaslavsky, 2004; Hadar & Hadass, 1981; Lockwood, Swinyard, & Caughman, 2015b). Our hope is that through investigating how students reason combinatorially, we may discover ways to foster productive thinking in combinatorics. Studies that have been conducted in this spirit include multiple reinvention studies (Lockwood, Swinyard & Caughman, 2015a; Lockwood & Shaub, 2016) where students generated basic counting principles and formulas. One such productive way of thinking that emerged from research is a set-oriented perspective (Lockwood, 2014), where students consider the set of outcomes as integral to the solving of counting problems. Other studies have developed and tested instructional interventions (Lockwood, Swinyard & Caughman, 2015b; Mamona-Downs & Downs, 2004). Our study contributes to this literature base by implementing generalization as a means to develop deep understanding of basic counting phenomena. We offer analysis of a task through which students construct the general formulas for basic counting operations such as arrangements and combinations. By analyzing their combinatorial understandings as they generalize, we learn more about the nature of students' combinatorial thought in these basic settings.

Theoretical Perspectives

Generalization

For purposes of describing students' activity as they generalize, we adopt Ellis' (2007a) taxonomy of generalizing activity. Ellis describes three main categories of generalizing actions, those of *relating, searching* and *extending*. *Relating* occurs when "a student creates a relation or makes a connection between two (or more) situations, problems, ideas, or objects" (p.235). Through *relating*, students organize mathematical phenomena they experience based on commonalities. The commonalities may be nuanced, and can take on different forms such as symbolic, structural, activity and more. The relationships formed may then become the source material for further generalizations.

Students also may engage in *searching*. *Searching* occurs when students perform "the same repeated action in an attempt to determine if an element of similarity will emerge" (p. 238). In this activity, students are seeking out regularity in the mathematical operations they perform. While *searching*, students may have some potential regularity they seek to verify, but also they may have yet to discover the regularity they hope to emerge via repeated action.

Finally, students generalize through *extending*. When *extending*, a student "not only notices a pattern or relationship of similarity, but then expands that pattern or relationship into a more general structure" (p.241). This may indeed be the activity most closely associated with generalization. While *extending*, students draw upon known mathematics and then apply them in a more abstract setting. This activity accounts for the learning of more abstract mathematics than previously encountered. This taxonomy allows us to organize students' activity as they generalize.

Combinatorial Reasoning

We also wish to describe the nuances of students' combinatorial understanding as they generalize. To do this, we utilize Lockwood's (2013) model for the different kinds of reasoning in combinatorics. Lockwood describes three unique and separate ways students reason about combinatorics problems. First, students may attend to the *formula* or *expression* of the combinatorial problem. Attention of this manner includes the symbolic form of the final answer, rather than the final numerical value. Students may also attend to the *counting processes* of a combinatorics problem. Students who reason via *counting processes* attend to the carrying out the process described by the problem. In doing so, they carry out the activity (either mentally or physically) to generate a solution to the problem. Finally, students might attend to the *sets of outcomes*. An outcome is a specific collection of the objects being counted. In considering this set, students attend to the particular structural organization of the outcome-set as a whole. This final way of reasoning is in line with Lockwood's (2014) *set-oriented perspective*, where the set of outcomes becomes a cornerstone of reasoning about any particular counting problem. Students reason in this way by viewing "atten[tion] to sets of outcomes as an intrinsic component of solving counting problems" (p.31). Further research has identified the productivity of attending to the sets of outcomes as well (Lockwood, 2013; Lockwood & Gibson, 2016).

Methods

The data for this report draws from two larger studies in which we investigated the nature of student generalization in combinatorial settings. To do this, we conducted one paired teaching experiment consisting of fifteen hour-long sessions followed by a design experiment consisting of nine ninety-minute sessions with four students. The students from both studies were recruited from vector calculus courses, and were selected from an initial set of applicants based on a selection interview process. Each of the students in these studies had not taken a discrete or combinatorics course before so that their activity and generalizations were indeed spontaneous rather than implementations of extant schemes.

This study reports on the generalizing activity of the students during the third session of each experiment. The goal of this session was to facilitate reflection on the students' prior activity from the previous sessions, culminating in the construction of general statements of certain combinatorial structures such as the permutation and combination. To do this, we presented the students with various problems they had previously solved either in the first two session or in the selection interviews and prompted them to separate the problems into groups. They were not given any specific instructions on how to group the problems so that distinctions they found relevant would be revealed. The problems included those whose solution methods were arrangement of n objects ($n!$), permutation of k objects from n objects $\left(\dfrac{n!}{(n-k)!}\right)$, selection of k objects from n objects $\left(\dfrac{n!}{(n-k)!k!}\right)$, and k repeated selections from a set of size n (n^k). Once the students agreed on the categories for the problems, they were asked to describe a general formulation of each category, and then to construct a general formula for the solution to each problem type. Both groups successfully categorized all problems into their respective four groups.

The sessions were video and audio recorded so that the records could later be reviewed for data analysis. The audio was also transcribed. Analysis consisted of reviewing the transcripts and the video files for episodes of generalizing activity. Relevant segments were further analyzed and coded according to Ellis' (2007a) framework and Lockwood's (2013) model.

Results and Discussion

The categorization task allowed students to both reflect on combinatorial structures and to meaningfully generalize from prior activity. In terms of generalization, we saw students relate and extend commonalities in the combinatorial situations with which they had previously engaged. Further, we saw students demonstrate a fluid ability to reason with and communicate across various components of Lockwood's (2013) model. Through this categorization, the students were able to generate multiple abstract combinatorial situations that demonstrated inherently different structures. Moreover, the students showed understanding of nuanced differences between the combinatorial structures. These understandings resulted from various generalizations rooted in reflection on activity. In this results section, we will both discuss the students' generalizing actions and combinatorial reasoning. This allows for a discussion of generalizing actions motivated by underlying combinatorial understanding.

The students engaged in meaningful relating activity while categorizing the different combinatorics problems. Through the relationships created, they were then able to make general statements reflecting the combinatorial structures. For instance, when first categorizing the problem types, Carson described a collection of problems involving selection with repetition that he had just arranged. Note that in this quotation he is describing multiple problems in front of him:

> This [referring to a specific collection of problems] is independent events. So, [first problem he describes] there are eight questions but the outcome of one doesn't affect the others. [Second problem he describes] There are six characters, but the outcome of one doesn't affect the others.

Here he was relating that each question described a combinatorial structure in which there was no dependence between selections. Indeed, while his language was in terms of outcomes, he described the outcomes not affecting other outcomes in the process. From this we infer Carson held a *process*-oriented perspective. Similarly, Josh then identified two more selection with repetition problems still not categorized:

> *Instructor*: . . . and why did those two go with those [the original collection Carson grouped together]?
>
> *Josh*: Those two also deal with independent events and finding all the possibilities in those events depend on something raised to some power.
>
> *Instructor*: Okay, okay, good.
>
> *Josh*: Like the number of choices that you have raised to the number of choices that you make.

Note here the difference between Josh and Carson's language as they engaged in the generalizing activity of relating. While Josh was responding to Carson's attention to independent events, Josh chose to identify these problems as similar according to the formula for the answer. According to Lockwood's (2013) model, Josh is attending to *formulas/expressions* while Carson is attending to *counting processes*. This diversity in combinatorial language was common during these discussions. Indeed, students often collaboratively generated the categories while appealing to individually different combinatorial details. For instance, while categorizing the same type of problem, the students in the teaching experiment had the following exchange:

Sanjeev: And then you want to paint 6 different houses on your block and there are 3 acceptable paint colors you can pick —
Rose: Would that one come down here? Because that would be —
Sanjeev: You have 6 houses and —
Rose: 3 to the power of 6?
Sanjeev: you have 3 different paint colors for each, yeah. So this [problem] would be this one [referring to the collection of selection with repetition problems]?

Notice that Sanjeev and Rose were attending to different components of Lockwood's (2013) model during this exchange. Sanjeev adopted a *process*-oriented perspective by attending to the process of picking paint colors. Rose, in turn, attended to the *formula* of the answer as a means of relating the houses problem to other selection with repetition problems she experienced. This further demonstrates the students' abilities to communicate and generalize across varying combinatorial language. While it may not be surprising that students are able to communicate efficiently while demonstrating various combinatorial understandings, we can witness a variety of cognitive material as the source for generalizations. For instance, Rose and Josh both demonstrated attention to common representation (*formula/expression*) of the solutions to the problems. As a contrast, Sanjeev and Carson demonstrated *process*-based relating amongst combinatorial situations. These students continued to attend to such nuances throughout the categorization task.

While there was, as noted, variety in the students' generalizations and combinatorial understandings throughout the task, we found a surprising uniformity of language pertaining to combinations. Indeed, all students demonstrated attention to the structure of the *sets of outcomes* when discussing combinations. The discussions about differentiating combinations from other combinatorial processes revolved around taking care not to count two similar outcomes as different. For instance, when separating the permutations and the combinations, Rose and Sanjeev said the following:

Rose: It's [referring to the collection of permutation problems they categorized]— it's how many — it's basically how many ways to put certain amount of items into fewer spots where 1, 2, 3 and 3, 2, 1 are different. And this [the collection of categorized combination problems] is how many ways you put a certain amount of things into fewer spots where 1, 2, 3 and 3, 2, 1 are the same.
Sanjeev: On these ones [referring also to the collection of permutation problems] you've got combinations [not referring to the combinatorial sense of the word. Literal combinations of outcomes]. So 1, 2, 3 - 3, 2, 1 would be different combinations. With this one [a combination problem], for example, if you have identical lollipops you can label them 1, 2, 3 or you can just label them 1, 1, 1. So 1, 2, 3 and 3, 2, 1 would be the exact same thing, because 1, 2 and 3 are all the same.

We note that while they mentioned permutations during this exchange as well as combinations, their previous discussion of permutations involved only discussing either the *formula* or the *process* involved in their construction. The distinction of making $\{1, 2, 3\}$ and $\{3, 2, 1\}$ the same was brought up as a means of separating the combination from permutation. Indeed, while there was a variety of combinatorial language used during categorization, students would always use set-based language when discussing combinations. We find attention to outcomes in this way as productive, as it allows for careful consideration of what is being counted. Further, it is interesting that combinations were the only problems in which there was uniformity in the combinatorial language that the students used.

As another example, we saw similar discussions of combinations emerge from the design experiment. Initially, when describing the difference between combinations and permutations, Ann-Marie remarked:

> Yeah, so in those two problems [a pair of combination problems] you divide by two factorials to cancel out the duplicate answers whereas in the other ones you don't have to do that.

Notice that her response also included *formula*-driven language. Indeed, Ann-Marie confessed that she primarily thought of the formula representation when thinking of the problem types. Ann-Marie made the distinction of "two factorials" in this case to contrast division by "one factorial" in the permutation group. What we see here is that within her formula driven remarks, she also used outcome-based language to describe the need for the extra division by a factorial. Also, later when explaining why the formula for $\binom{n}{k}$ adds on a division by $k!$, Aaron explained:

> Well, because you're trying to get rid of all the combinations that you're not looking for that you can make out of those three slots because they're all the same. So, that just accounts for it.

Indeed, most descriptions of combinations involved *outcome* - based language so that they could be differentiated from permutations. Often, the design experiment students described "dividing by redundancies" when performing combinations. It is interesting that among the students we worked with, combinations were uniformly a source of *outcome* - based language. Returning to the teaching experiment, we see Rose also using *outcome* - based language when describing why a subset selection problem is grouped with other combinations. After negotiating the particulars of the problem involving finding subsets of a set of numbers, Rose said the following:

Rose: and if that was the case then we'd want to put it over into this group [the collection of combination problems].

Int: Okay. And how come?

Rose: Because now you don't want — you just want unique combinations. And if you're getting rid of all the — the repeated subsets, then you're just finding the unique combinations.

Here, we see Rose clarified that the desired outcomes were indeed "unique combinations". The uniqueness was generated by getting rid of repeated subsets, which indeed would emerge from a standard permutation. Thus, we see that Rose diverged from her typical *formula*-centered language to attend to unique outcomes. Finally, when also discussing the subset selection problem, the design experiment students had the following exchange:

Carson: . . . and these [their initial collection of permutation problems] you're arranging a given number of things in smaller number of spaces than there are things.

Josh: Is this one [the subsets problem] really the same as the others though because you're only looking for the four number set?

Carson: So, every number in the four limit subset is unique, right? So, there's no repeated numbers.

Josh: There can be repeated numbers.

Ann-Marie: But like zero, one, two would be the same as one, zero, two.

Carson: Right and you can't have zero, zero, zero.

Josh: Oh, yeah.

Carson: So, that would be an arrangement one as well just with the caveat that there are only four of them.

Here we see two types of outcome - differentiation occurring. We see the students describing that ordering the numbers in the subset should not create a different outcome. This is consistent with the set-oriented perspective the students took on combinations. We also see Carson noting that an outcome cannot have multiples of a number in the subset. While this is not unique to combinations, it is another example of *set-oriented* language. The above discussion naturally centered around whether or not certain outcomes would be considered as distinct. Indeed, the students in both groups consistently attended to the set of outcomes while discussing combinations.

While much of the above discussions centered around the activity of relating, we also saw students engage in extending. The students were prompted to generate statements and formulas that reflected the categories they had created. The following statement the teaching experiment students wrote for the collection of combinations further reflects the *outcome*-centered underpinnings of their generalizations. Rose and Sanjeev produced the following characterization of combination problems:

3) $\uparrow_2$ *... and divide by the factorial of the given spots to delete repeated sequences because any arrangement of the same given elements is considered the same combination.*

Note that the arrow marked with a 2 at the beginning is referring to their previous statement of a permutation process. This characterization suggests that the structure of a combination involved constructing a permutation followed by the further operation of division as described above. We bring attention to their *outcome*-oriented justification for their addition to the permutation. This further demonstrates that their understanding of a general combination process involves accounting for multiple arrangements of a particular outcome. We again note that such distinctions are productive, and allow for deeper understanding of the combinatorial objects.

Conclusions

We see that the categorization task allowed the students to generalize their prior work on individual counting problems into more general contexts in which different combinatorial structures could be illuminated. The students productively engaged in relating and extending, both activities underpinned by the nuances of the combinatorial settings, as described by Lockwood's (2013) model. We saw meaningful generalizations being underpinned by all three aspects of Lockwood's model. Moreover, there was a uniformly *set-oriented* approach to generalizing combinations. Further, such distinctions between permutations and combinations demonstrated productive understandings of the combinatorial objects. Such a perspective allows for specific criteria with which students can evaluate whether a combination or permutation applies to a counting situation. Thus, we see that through engagement in categorizing and reflecting on prior work, students meaningfully generalized while gaining a better understanding of the combinatorial objects.

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