Modus Tollens in Modeling

Jennifer A. Czocher
Texas State University

Jenna K. Tague
Fresno State University

The purpose of this paper is to present a case study of a mathematics major exhibiting logical reasoning to validate her mathematical model. The case study demonstrates how constructing a mathematical model can be construed as making an argument for its validity.

Keywords: mathematical modeling, mathematical argumentation, mathematics majors

There is a plurality of views and foci on teaching and learning mathematical modeling (Cai et al., 2014). The cognitive view on modeling has focused on how the modeler transforms the nonmathematical problem into a mathematical one (Kaiser & Sriraman, 2006). Several frameworks have been introduced to capture this transformation and allow it to be finely analyzed according to modeling competencies (Blum & Leiß, 2007), prior mathematical knowledge (e.g., Stillman, 2000), prior real-world knowledge (e.g., Czocher, under review), and theories of metacognition arising from problem solving (e.g., Galbraith & Stillman, 2006; Panaoura, Gagatsis, & Demetriou, 2009). While prior analyses have explained a great deal of how productive and unproductive moves within the modeling process may be characterized, they are limited to examining only specific modeler moves *within* the modeling process. With respect to mathematical reasoning, these frameworks are limited to examining only the mathematics the modeler uses to set up, analyze, compute, or solve the resulting model which can usually be explained in terms of the mathematics content intended by the task writer. That is, these frameworks do not allow documentation of validating the model if the means to do so fall outside of the expected mathematics or modeling context. This paper presents a case study of how an individual might use logic to guide her use of mathematical content knowledge. We follow with a discussion of why students' logic might have been overlooked in other frameworks and then discuss why an alternative lens for examining modeling behavior, especially of more advanced students, is promising for shedding light on similarities among modeling, problem solving, and proving.

Background

From a cognitive perspective, studying mathematical modeling means attending to the mathematical thinking that produces the model (Borromeo Ferri, 2007). Mathematical modeling is viewed as a process that transforms a question about the real world into a mathematical problem to solve (Frejd, 2013). The answer to the mathematical problem is then interpreted as a solution to the real world problem. This process is often represented as a cycle (e.g., Blum & Leiß, 2007), which is summarized in Table 1. Much of the research on modeling from the cognitive perspective focuses on the simplifying/structuring phase (identifying variables, making assumptions) and on the mathematizing phase (introducing conventional representational systems). Comparatively less research has focused on validating, which involves checking that the mathematical model is representative of the situation and that it is correctly analyzed (solved) mathematically. Validating is challenging to study because of how the modeler perceives and resolves cognitive conflict between their expectations of their model (e.g., predictions) and outcomes (e.g., empirical observations) (Czocher, 2014, 2015). Students may respond to cognitive conflicts in less-than-ideal ways (Goos, 2002). They may fail to notice that something is amiss, perceive difficulties that do not exist, provide an inadequate response, or even change the problem to suit their readily-available knowledge (Goos, 1998, 2002; Stillman, 2011). Indeed, some have observed that validating is a "uniform shortcoming" of students' mathematical modeling because they do not *reflect* to improve their models at all (e.g., Blum & Leiß, 2007). However, some small amount of work has revealed that engineering undergraduates do engage in validating their models, typically through techniques like dimensional analysis, checking special and limiting cases, making comparisons to empirical results, and relying on number sense (Czocher, 2013). On the other hand, mathematics majors' conditional reasoning has been documented, particularly as it relates to comprehending an argument (Alcock, Bailey, Inglis, & Docherty, 2014). The following analysis is an effort to begin to document and understand the reasoning mathematics majors use to validate their mathematical modeling work.

Table 1 Indicators from the observational rubric to identify subprocesses in the MMC (Czocher, 2016)

Modeling Subprocess	Definition	Examples of Observed Student Activity
Understanding	Forming an initial idea about what the problem is asking	Reading the task Clarifying what needs to be accomplished
Simplifying & structuring	Identify critical components of the mathematical model (i.e., create an idealized view of the problem)	Listing assumptions or specifying conditions Identifying variables, parameters, or constants Operationalizing quantities or relationships
Mathematizing	Represent the idealized model mathematically	Writing or speaking mathematical representations of ideas (e.g., symbols, equations, graphs, tables, .)
Working mathematically	Mathematical analysis	Explicit algebraic or arithmetic manipulations Making inferences and deductions without reference to nonmathematical knowledge Changing mathematical reprsentation
Interpreting	Recontextualizing the mathematical result	Speaking about the result in context of the problem or referring to units Considering if the result answers the question posed
Validating	Verifying results against constraints	Implicit or explicit statements about the reasonableness of the answer/representation Checking extreme or special cases of variables, parameters, relationships, etc. Dimensional analysis of units

Methods

Qualitative data were generated via an individual task-based interview (Clement, 2000; Goldin, 2000). The tasks were a variety of modeling and application problems drawn from previous research (e.g., Ärlebäck, 2009; Czocher, 2016; Schoenfeld, 1982; Swetz & Hartzler, 1991). The 10 tasks were sufficiently open to allow participants to select their own variables, assumptions, and solution techniques. The purpose of the interviews was to elicit participants' mathematical thinking as they engaged in mathematical modeling; the interviewer did not guide participants to a solution, but intervened only to request clarification or to extend the task. In this paper, we focus on a single case to illustrate a mathematics major's reasoning on a conventional word problem. The case is illustrative of a mathematics major using clearly outlined logic despite arriving at a wrong answer. The data are presented and analyzed as a narrative, a "spoken or written text giving an account of an event/action or series of events/actions, chronologically connected" (Czarniawska, 2004, p. 17). To do so, we view the interview participant, Safi, as presenting an account her series of decisions during mathematical modeling.

Safi was a senior mathematics major at a large southwestern university. She was enrolled in a vector calculus course and stated that her favorite subjects thus far were "linear algebra, hands down, and differential equations." She was nearing completion of her mathematics requirements and was seeking secondary teacher certification. Safi had completed her first classroom internship in geometry at a local high school, but stated a preference for teaching algebra. The following semester, before graduation, she was scheduled to do her student teaching in an algebra 2 classroom. Safi did not describe herself as good at mathematics. She said, "since being here [at university] I have struggled with like my math classes and everything but I've worked really hard to get even like the C's I have gotten." Safi valued the hard work she put into her classes, which fueled her drive to be a teacher, despite the fact that the higher level mathematics courses she didn't "really see being useful, like the proof classes." She elaborated that the content of the proof classes would not be something she used in her high school classes but that "maybe the different way of thinking" would be useful.

Below, we present Safi's work on the Turkeys & Goats problem (Czocher & Maldonado, 2015) and analyze it in terms of the correctness of her response, her engagement in mathematical modeling, and the reasoning she used to arrive at her conclusions. The problem was: *A nearby farm raises turkeys and goats. In the morning, the farmer counts 48 heads and 134 legs among the animals on the farm. How many goats and how many turkeys does he have?* The problem is a word problem (see Gerofsky, 1996) that is ubiquitous in secondary school algebra textbooks and on standardized tests. The answer, 19 goats and 29 turkeys, can be obtained in a variety of ways including setting up a system of two equations in two unknowns. Because of Safi's mathematical training and recent experiences in mathematics pedagogy, the task was well within her capabilities. In order to analyze Safi's engagement in modeling, the observational rubric from Table 1 was applied. When Safi was observed, in speech or writing, to be carrying out one of the activities in the right-most column, her activity was coded with the corresponding modeling subprocess from the left-most column.

Presentation of Safi's Reasoning

Safi began by reading the Turkeys & Goats problem aloud [understanding]. She then emphasized some information, "48 heads and 134 legs" which she repeated aloud and wrote down [simplifying/structuring]. She then explicitly identified what needed to be accomplished, "and then they're asking how many of each animal" [understanding]. She narrated her reasoning, "48 heads means he has 48 animals in total because he wouldn't have more heads than animals because that wouldn't make sense." In this statement, Safi engaged in both simplifying/structuring because she established the condition that 48 heads means 48 animals in total and validating because she was evaluating its sensibility. To carry out her validating, she used counterfactual reasoning (reasoning from a situation that doesn't or can't exist) to set up and evaluate a brief propositional logic argument positing a one-to-one correspondence between heads and animals. She assigned the variable x to the number of turkeys and the variable y to the number of goats [mathematizing]. She then wrote the two equations $x + y = 48$ and $2x + 4y = 134$ [mathematizing], checking that "two legs per turkey will give you the amount of turkey" legs [validating]. Safi used elimination method to solve the system [working mathematically]. She obtained $y = 19$ which she interpreted to mean "there should be 19 goats" [interpreting]. Then to obtain the number of turkeys, she computed $48 - 19 = 27$ using the standard algorithm [working mathematically]. She wrote 27 turkeys [interpreting]. To check her work, she used standard algorithms to compute $2 \times 27 + 4 \times 19 = 134$ [validating]. She obtained 130 for the left hand side. She asked "Am I allowed to ask you the amount of turkey legs?"

Safi had arrived at a contradiction: her solution 19 goats and 27 turkeys did not yield the same number of legs set by the conditions in the problem statement. Her first recourse was not to doubt her computation but to doubt whether turkeys had 2 legs. The interviewer followed up by exploring whether 2 legs per turkey was a logical antecedent or logical consequence of 19 goats.

Safi: I solved it with turkeys having two legs, but I am short 4 legs.

Interviewer: You're short 4 legs. And you are certain that they are turkey legs?

Safi: No. But if turkeys have 2, then I am not sure. Well, 'cause I solved it to where goats have the 19, there were 19 goats.

Interviewer: Okay, so given that turkeys have 2 legs, there must be 19 goats. Is that what you're saying?

Safi: Yeas. Oh wait, wait wait. But okay wait. The goats here…they have 38 legs, and then [[talks quietly then laughs]]. Yeah, so given that turkeys have 2 legs, there should be 19 goats.

Safi continued this chain of logical reasoning to argue that given that turkeys have 2 legs, there must be 19 goats, so there have to be 27 turkeys. There can't be 27 turkeys because $27 \times 2 = 54$, meaning just turkeys alone would have 54 legs. Given that there are 19 goats and goats have 4 legs, they would have 76 legs. Altogether there would be 130 legs, which is too few legs. Safi "called into question" the assumption that turkeys have 2 legs.

After a brief discussion about why Safi had chosen to use the operations + and × where and how she did to set up her system of equations, the interviewer extended the problem. Instead of Turkeys and Goats, the interviewer posed a problem in which the farm had pigs and goats, with 48 heads and 134 legs. The resulting system of equations was inconsistent. Safi set up the equations, solved them via elimination and obtained the result "0 equals negative." She interpreted it to mean that there could be no goats and therefore there were 48 pigs.

Safi: But then if you have 48 pigs, each pig should have 4 legs, which would mean 192 legs. But there is 134. So that's, that's not accurate.

Interviewer: Which isn't accurate? The 192 or the 134 or something else?

Safi: Well, if you're paying attention to the heads, like it depends on what you're looking for, if you're looking at the heads. Then the legs, the 134 doesn't make sense because you have more, you realistically have 192 legs here with 48 pigs. And it says you only have 134. So that's not enough to complete your farm [[laughs]].

Interviewer: So, when you, I noticed you like put in this adjective there, you "realistically" how would you have 192 legs? What did you mean by that?

Safi: So, we mean, you could have pigs missing legs. Um, 'cause they don't need four legs to be able to live so if you take out some, we mean we guess you could get to 134. But realistically, if they all have 4 legs then that's how many you would have, you have 192.

In follow-up questioning, Safi revealed that she noticed that both versions of the problem were similar to those she had seen in "algebra and algebra 2 and linear algebra" and so she was readily able to set up the system of equations and "in order to solve for each variable you usually just do any process that you can," though she did not use the vocabulary of linear algebra to seek solutions or explain the lack of solutions to each system.

Discussion and Conclusions

Safi did not arrive at the correct solution for either the Turkeys & Goats problem, due to the arithmetic error $48 - 19 = 27$. She also did not realize that there was no solution to the system of equations she derived for the extension problem although she recognized a contradiction for the number of legs required. However, in both versions of the task she did engage in the cognitive activities underlying mathematical modeling (as suggested by the observational rubric) and she did reach conclusions that were logically consistent with the information she gleaned from the task statement. On the surface, it seems unreasonable that Safi would doubt a basic fact like *turkeys have two legs*. Closer inspection reveals that it is a logical consequence of an argument she constructed to validate her model (the system of two equations in two unknowns) and its prediction (the number of turkeys and goats on the farm). Table 2 shows her argument's structure mapped to propositional logic:

Table 2. Safi's logical argument

1.	There are 134 legs on the farm (premise)	5.	There are 27 turkeys (3, 4)
2.	Turkeys have two legs (premise)	6.	There are 130 legs on the farm (3, 4, 5)
3.	The system $x + y = 48$, $2x + 4y = 134$ describes the number of animals on the farm (1, 2)	7.	Contradiction (1, 6).
4.	There are 19 goats (3)	8.	Reject (2).

Safi checked her by-hand computations twice to be sure that (4) and (5) turned out correct (committing the same mental arithmetic error each time). Her only course of action, logically, is to reject one of the two premises upon which (3) stands. Since (1) is given in the problem, she must reject (2). Her spoken argument can be reduced to the form of modus tollens: If turkeys have two legs, then there are 130 legs on the farm. There are not 130 legs on the farm. Therefore, turkeys do not have two legs (she expressed an equivalent summary verbally). She displayed similar reasoning patterns on the extension to the pigs and goats problem.

What is interesting about Safi's response is not that she is a math major who is a preservice teacher who got a routine word problem incorrect (which is the sort of result documented in the past); rather, the novelty of Safi's work is how she used logical reasoning from her advanced mathematics courses to make sense of and support her conclusions about the validity of the mathematical model she constructed. Students' untrained reasoning may be incompatible with mathematical logic and students' application of logical structure largely depends on the semantic context (Dawkins & Cook, 2017). Safi was a student trained in logic and mathematical reasoning with knowledge of the semantic context. Deconstructing Safi's response in terms of a first-order propositional logic revealed how it supported her interpretation and validation of her model, and opens questions about whether students' mathematical thinking during modeling may be productively analyzed according to argumentation models (e.g., Toulmin schemes). Her responses also suggest that such lenses might reveal insights into the interaction between content knowledge and mathematical modeling. Further, Safi's use of logic shows that mathematics majors may not all have the same validating techniques at their disposal as engineering or science majors, implying that caution must be exercised when generalizing conclusions about modeling behavior among any of these populations (Czocher, 2013). In particular, if mathematics majors are using the skills and patterns of reasoning that they learn in advanced proof-based courses in other domains it raises new questions about the natures of mathematical modeling, problem solving, and proving and what characteristics they may share. Scholars in either area must be cautious of overlooking kinds of reasoning not typically linked to the domain of inquiry. For these reasons, further work needs to be done to document what validation processes students are likely to bring from various backgrounds and how they contribute to the students' mathematical modeling processes.

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