Identifying Subtleties in Preservice Secondary Mathematics Teachers' Distinctions Between Functions and Equations

James A. Mendoza Álvarez[1] Theresa Jorgensen[1] Kathryn Rhoads[1]
[1]The University of Texas at Arlington

For more than thirty years, the secondary school mathematics curriculum has seen a shift to functions-based approaches to algebra. Advancing comprehension of the equals sign as an equivalence relation is critical for beginning algebra students studying equations, and developing understanding of functions is foundational as a gateway to courses required of science, technology, engineering, and mathematics majors. This study explores the ways in which mathematics majors seeking secondary mathematics teaching certification distinguish between the concepts of function and equation. Participants (n=24) completed a ten-item pre- and post-assessment on functions and equations. Open coding techniques were used to identify emerging categories that describe participants' distinctions between the concepts. After a mathematics course experience with an eight-week unit on functions, the participants' concept image for functions focused primarily on input and output whereas their concept image for equations centered broadly on the equivalence of two quantities.

Keywords: preservice secondary mathematics teacher preparation, function, equation

The topic of *functions* has been well-documented in the research literature as "difficult for students to learn, challenging to teach, and critical for students' success as learners and in their future lives and careers" (Cooney, Beckmann, & Lloyd, 2010, p. v). Students in the United States are commonly introduced to functions in secondary school (National Governors Association Center for Best Practices and Council of Chief State School Officers [CCSSM], 2010). Given the importance and difficulty of functions, it is essential that secondary mathematics teachers have the depth and breadth of understanding necessary to teach this critical topic (e.g., Stacey, 2008), and undergraduate studies offer an opportunity for teachers to build a profound understanding of functions.

Part of a profound understanding of function includes a clear articulation of the differences between functions and equations. Although there are ways to relate the topics of *function* and *equation*, they are sometimes inappropriately conflated by students and teachers alike. In this study, we investigate the following research questions:

1) How do preservice secondary mathematics teachers distinguish between functions and equations?
2) What subtleties exist in preservice teachers' distinctions?

Theoretical Framework

To frame this study, we draw on Tall and Vinner's (1981) theory of *concept image* and *concept definition*. Throughout their school studies, preservice secondary math teachers develop a concept image of the topic of functions, which includes "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). A teacher's concept image of function (for example) may be well-developed and align with the formal definition of function, or their concept image may be fragmented, incomplete, or misaligned with the formal definition. Teachers may also have a personal concept definition for function—that is, the words the teacher uses to define function. A teacher's personal concept definition may reflect their concept image, or it may be misaligned from their concept image. At the same time, one's personal concept definition may align with (or be a memorized recitation of) the formal concept definition in the mathematical community, or it may be inconsistent with the formal definition.

Alignment between concept image and the formal concept definition is important because conflicts between these two may cause difficulties in students' learning (Tall & Vinner, 1981). In addition, concept images or personal concept definitions that are misaligned with the formal definition may cause students to think that the formal definition is "inoperative and superfluous" (Tall & Vinner, 1981, p. 184). Alignment between concept image and concept definition is especially important for teachers who are guiding students' learning of the concept. In this study, we investigate preservice secondary teachers' concept images and personal concept definitions of function and equation.

Research Literature

Throughout high school and undergraduate mathematics, students are accustomed to working with functions which can be defined by algebraic formulas, and students often use formulas to identify the functions they discuss (Cooney et al., 2010). Formulas for functions are especially useful in calculus, and undergraduate courses such as calculus can reinforce students' concept image of functions being defined by formula. In fact, students' conceptions of functions can be limited by thinking of them as defined by formulas. For example, Even (1993) surveyed 152 preservice secondary mathematics teachers about functions, and ten additional preservice teachers were interviewed. Many of these preservice teachers thought that functions could always be represented by an algebraic formula. Similarly, using questionnaires with 30 secondary teachers, Hitt (1998) reported that many teachers believed that functions could always be represented by a single algebraic expression, and Carlson (1998) reported the same finding for students who earned A's in College Algebra.

Secondary school curriculum emphasizes that zeros of a function *f* are the solutions to the equation $f(x) = 0$ (CCSSM, 2010). Although this connection is valuable, students sometimes muddy this relationship. For example, in a study with students earning A's in College Algebra, Carlson (1998) found that these top-performing students "do not make a distinction between the zeros of functions and solutions to equations" (p. 141). In a 1999 study, Carlson also reported that second-semester calculus students had similar confusions between solutions to equations and zeros of functions.

To further complicate matters, in high school as well as undergraduate mathematics, a formula such as $f(x) = 3x + 2$ is sometimes referred to as *the equation for the function* f or *the defining equation for the function* f. Perhaps perpetuated by this terminology, many preservice teachers have some incorrect conceptions about the relationships between functions and equations. For example, in Even's (1993) study, some preservice teachers provided definitions of function in which they claimed a function *was* an equation or expression. Breidenbach, Dubinsky, Hawks, and Nichols (1992) found that some preservice mathematics teachers described a function as "a mathematical equation with variables" (p. 252). Not surprisingly, Chazan & Yerulshamy (2003) documented that learners also have difficulty in distinguishing between functions and equations.

Methodology

This study was conducted at a large, urban university in the southwestern United States with an on-campus student enrollment larger than 37,000 students. Due to the large enrollment (greater than 25% of the student body) of Hispanic students, the university carries a US Department of Education Hispanic Serving Institution designation. In addition, the university is described as one of the most diverse national universities in the United States.

The population for this study was preservice secondary mathematics students who were enrolled in a second-year mathematics course in the fall semester of 2016. The course, Functions and Modeling, is a required course for mathematics majors seeking secondary mathematics teaching certification. The intent of the course, which carries a second-semester calculus prerequisite, is to deepen preservice secondary mathematics teachers' experiences with the mathematics that they will teach, immerse them in an inquiry-based learning environment, and develop a profound understanding of important concepts for secondary school mathematics. In the 15-week fall 2016 semester, approximately eight weeks of the course focused on functions and patterns, four weeks on regression and modeling, and three weeks on various topics such as parametric equations, polar coordinates, vectors, and the geometry of the complex numbers.

Thirty students (17 females and 13 males) were enrolled in the course and 24 students participated in the research study. The overall student population enrolled in the university's science and mathematics secondary teacher certification program is 41% Hispanic, 38% White, 14% Asian, and 7% Black.

A written instrument consisting of ten items (and corresponding sub-items) targeting the students' understanding of function and equation was used as a pre- and post-assessment. The items on the assessment required the preservice teachers to explain their reasoning and, where appropriate, provide multiple representations. The assessment took the students approximately one hour to complete. This study examines student responses to two of the assessment questions.

- "Can the terms *function* and *equation* ever be used interchangeably? Why or Why not?"
- "If a student in Algebra I asked you to explain the difference(s) between a function and an equation, what would be your response?"

The pre-assessment was administered during the first week of the course. The post-assessment was completed after the course final exam. Qualitative methods were used to analyze the written responses from the assessments. Participant responses were systematically coded by elements in their explanations and by themes that emerged in the data relevant to their descriptions comparing the concepts of function and equation.

In the analysis of the pre-and post-assessments, we used principles of the grounded theory method (Strauss and Corbin 1990), allowing the data to be coded through the lens of emerging themes. The data were then grouped into similar conceptual themes characterize the preservice teachers' descriptions of contrasting function and equation.

Results

Participant responses to "Can the terms *function* and *equation* ever be used interchangeably? Why or Why not?" on the pretest were coded as ambiguous (AMB), relationship/both (RLB), non-answer (NAN), some equations are not functions (ENF), some or all equations are functions (SEF), and some or all functions are equations (SFE) (see Table 1).

On the posttest, the new codes relationship vs. equivalence (RVE) and definition (DEF) arose from the posttest data. Responses that rejected interchangeability by mentioning the difference in the way the terms are defined were coded DEF. For example, "no, their definitions are not the same" was coded DEF. Responses that claim equations assert equivalence between two quantities but functions depict an input-output relationship were coded RVE. The codes RLB and ENF did not appear while there were 2 AMB, 6 NAN, 3 SEF, 8 SFE, 3 RVE, and 2 DEF.

Table 1. Codes arising from the Pretest interchangeability question and their frequency.

Code	Description	Selected Response	Freq. (n=24)
AMB	Ambiguous response that does not offer reasons.	“not always interchangeable; depends on how it is written”	3
RLB	Claim that both express a relationship.	“Yes, because they both describe a relationship between variables…”	4
NAN	Non-sensical or non-mathematical response.	“No! hmm maybe…wow, you’ve got me stumped…”	6
ENF	Asserts that not all equations are functions.	“no, because not every equation is a function”	5
SEF	Asserts that some or all equations are functions.	“they can be interchanged sometimes there are equation that describes functions, but not always”	2
SFE	Asserts that some or all functions are equations.	“they can be, for example the function of x (f(x)) can be displayed as y”	4

Three of the six participants who provided an NAN-coded response on the pretest also provided an NAN-coded response on the posttest. Three other participants’ response codes remained the same from pretest to posttest—two SFE responses and one SEF response. The five ENF-coded responses on the pretest provided two NAN-coded responses, two SFE, and one SEF-coded response on the posttest.

Participant responses to “If a student in Algebra I asked you to explain the difference(s) between a function and an equation, what would be your response?" on the pretest were coded as NAN, SEF, input-output (IO) with sub codes equation equivalence (EE) or equation number specific (NS), and relationship (RL) with sub codes equation equivalence (EE) and equation number-specific (NS) (see Table 2).

Table 2. Codes arising from the Pretest Algebra I student question and their frequency.

Code	Description	Selected Response(s)	Freq. (n=24)
IO	Refers to input-output or independent-dependent variables for functions and	-IOEE: “Function: independent variable dictates the value of the dependent variable. Equation: something equals something else.”	6
IOEE	-equations asserting equivalence of two quantities, or		
IONS	-equations as specific situations when numbers are used.	-IONS: “…equation may just involve solving for one variables [sic] given a number…”	2

RL RLEE RLNS	Refers to mapping or relationship between variables and -equations asserting equivalence of two quantities, or -equations as specific situations when numbers are used.	-RLEE: “An [sic] function assigns all the elements in set x to set y simultaneously. While an equation does not assign it simply equates. -RLNS: “…equation takes that relationship and puts numbers in it…”	2 5
NAN	Non-sensical or non-mathematical response.	“I’m not sure I’d have the best response right now.”	7
SEF	Asserts that some or all equations are functions.	“An equation can be a type of function…”	2

On the posttest, the new code representation vs. equation equivalence (RPEE) was needed to code answers that referred to a representation to distinguish between function and equation; for example, “…a function has to pass the vertical line test.” The sub codes IONS and RLNS disappeared while there were 2 RPEE, 7 IOEE, 4 NAN, 7 RLEE, and 4 SEF. The six of the seven participants with NAN responses on the pretest coded for IOEE or RLEE on the posttest with one receiving a SEF code.

Discussion

Although participants completed several inquiry-based lessons that focused on precise definitions of functions and equations as well as several lessons using functions to model data, only one participant, on the “Algebra I student question,” used the terms domain and codomain when attempting to make an equation-function distinction. Somewhat akin to Carlson’s (1998) findings, no participants attempted to contrast equations and functions by referring to solution sets or domain and range, respectively. As in Even (1993), the use of the equal sign when defining a function with an algebraic expression may explain why 8 of 48 responses—aggregating the responses to both questions—still assert that some functions are equations.

The prevailing concept image for function entailed input-output or the idea that a function establishes a relationship between inputs and outputs, regardless if their description of an equation also used the idea of a relationship between quantities. Possibly a result of a lesson specifically focusing on the role of the equal sign in defining, equivalence, and computation may have influenced a shift from number-specific responses about equations to 14 of 24 responses that gave a mostly-correct equation concept definition.

The purpose of this study is to further investigate the subtleties in preservice secondary mathematics teachers’ conceptual distinctions between function and equations. Further input from researchers is needed regarding developing interview protocols, alternative assessment questions, and ways to interpret the data that inform curriculum development and instruction.

Acknowledgements

This research is based upon work partially supported by the National Science Foundation (NSF) under grant number DUE-1612380. Any opinions, findings, conclusions or recommendations are those of the authors and do not necessarily reflect the views of the NSF. The authors would also like to thank Janessa M. Beach for research assistance.

References

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics*, *23*, 247–85.

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. *Research in collegiate mathematics education*, *3*, 114-62.

Carlson, M. (1999). *A study of second semester calculus students' function concepts*. Unpublished manuscript.

Chazan, D., & Yerushalmy, M. (2003). On appreciating the cognitive complexity of school algebra: Research on algebra learning and directions of curricular change. In J. Kilpatrick, W.G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 123-135). Reston, VA: National Council of Teachers of Mathematics.

Cooney, T. J., Beckmann, S., & Lloyd, G. M. (2010). *Developing essential understanding of functions for teaching mathematics in grades 9-12.* Reston, VA: National Council of Teachers of Mathematics.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education*, *24*, 94–116. doi:10.2307/749215

Hitt, F. (1998). Difficulties in the articulation of different representations linked to the concept of function. *Journal of Mathematical Behavior, 17*(1), 123-134.

National Governors Association Center for Best Practices and Council of Chief State School Officers. *Common Core State Standards for Mathematics.* Washington, DC: Authors, 2010.

Strauss, A., & Corbin, J. (1990). Basics of qualitative research: Grounded theory procedures and techniques. Newbury Park, CA: Sage.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.