Drawing on Three Fields of Education Research to Frame the Development of Digital Games for Inquiry-Oriented Linear Algebra

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Abstract: Demands in undergraduate education are shifting to reach larger student populations - especially learners beyond the brick-and-mortar classroom - which has led to more pressing demands to incorporate technologies that afford such learners access to high-quality, research-based, digital instructional materials. In this article, we explore three theoretical perspectives that inform the development of such instructional materials. In our team's efforts to develop a game-based learning applet for an existing inquiry-oriented curriculum, we have sought to theoretically frame our approach so that we can draw on the corpus of researcher knowledge from multiple disciplines. Accordingly, we will discuss three bodies of literature – realistic mathematics education's [RME's] approach to curriculum development, inquiry-oriented instruction and inquiry-based learning [IO/IBL], and game-based learning [GBL] - and draw on parallels across the three in order to form a coherent approach to developing digital games that draw on expertise in each field.

Keywords: Realistic Mathematics Education, Inquiry-Oriented Teaching, Inquiry-Based Learning, Game Based Learning, Linear Algebra

Introduction

A number of researchers in undergraduate mathematics education have developed curricula that draw on the curriculum design principles of Realistic Mathematics Education (RME) and are intended to be implemented using an inquiry-oriented (IO) approach (e.g., Larson, Johnson, & Bartlo, 2013 (abstract algebra); Rasmussen et al., 2006 (differential equations); Wawro et al., 2012 (linear algebra)). IO curricula fall within the broader spectrum of Inquiry-Based Learning (IBL) approaches that focus on student centered learning through exploration and engagement (Ernst, Hodge, & Yoshinobu, 2017) facilitated by an instructor's interest in and use of student thinking (Rasmussen, Marrongelle, Kwon, & Hodge, in press). For the purpose of this paper we will give examples from an IO curriculum, but also use quotes and references from the more general IBL literature.

In our current project we are exploring the extent to which technology can help mathematics educators extend inquiry-oriented (IO) curricula into learning contexts that are less conducive to inquiry-oriented approaches. Game Based Learning (GBL) provides a reasonable approach to addressing the constraints that large class sizes or non-co-located learning place on instructors' implementation of IO curricula. GBL studies show a clear relation between games and learning as games provide a meaningful platform for large numbers of students to engage, participate, and guide their learning with proper and timely feedback (Barab, Gresalfi, & Ingram-Goble, 2010; Gee, 2003; Hamari et al., 2016; Rosenheck, Gordon-Messer, Clarke-Midura, & Klopfer, 2016). However, despite advances in technology and policy initiatives that support development of active learning and the incorporation of technology in classrooms, few digital games exist at the undergraduate level that explicitly incorporate a research-based curriculum. In this paper, we explore the three theoretical perspectives of RME, IO/IBL instruction, and GBL in order to identify the ways in which the three perspectives align and might contribute to the development of digital media that incorporate knowledge and practices gained from each perspective.

We begin with a discussion of each of the three theoretical framings illustrated with specific examples. For the first two framings we describe a task sequence and strategies for implementing that task sequence that come out of the Inquiry Oriented Linear Algebra (IOLA) curriculum. For the third framing, we provide a brief outline of a mathematics game, *Rolly's Adventure*, developed by the third author, who drew on GBL principles in her game design. We then draw on each of these examples to demonstrate how aspects of RME, IO/IBL instruction and GBL align with each other and to point out a few ways that RME and IO/IBL might be used to inform design of future games, especially as we, the authors, move towards the development of a new digital game rooted in the existing IOLA curricular materials.

Realistic Mathematics Education and Inquiry-Oriented Linear Algebra (IOLA)

Realistic Mathematics Education is a curriculum design theory rooted in the perspective that mathematics is a human activity. Accordingly, RME-based curricula focus on engaging students in activities that lend themselves to the development of more formal mathematics. Researchers rely on several design heuristics to guide the development of RME-based curricula (Gravemeijer, 1999; Rasmussen & Blumenfeld, 2007; Zandieh & Rasmussen, 2010). For instance, researchers often focus on the historical development of the concept intended to be taught so that the curriculum supports students' guided reinvention of the mathematics. In this paper, we focus on Gravemeijer's (1999) four levels of activity to show how curricula might reflect the design theory. **Situational activity** involves students' work on mathematical goals in experientially real settings. **Referential activity** involves models-of that refer to physical and mental activity in the original setting. **General activity** involves models-for that facilitate a focus on interpretations and solutions independent of the original task setting. Finally, **formal activity** involves students reasoning in ways that reflect the emergence of a new mathematical reality and no longer require prior models-for activity.

The IOLA curriculum (http://iola.math.vt.edu) draws on RME instructional design heuristics to guide students through various levels of activity and reflection on that activity to leverage their informal, intuitive knowledge into more general and formal mathematics (Wawro, Rasmussen, Zandieh, & Larson, 2013). The first unit of the curriculum, referred to as the Magic Carpet Ride (MCR) sequence, serves as our example of RME instructional design (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012). As stated, **situational activity** involves students working toward mathematical goals in an experientially real setting. The first task of the MCR sequence serves to engage students in **situational activity** by asking them to investigate whether it is possible to reach a specific location with two modes of transportation: a magic carpet that, when ridden forward for a single hour, results in a displacement of 1 mile East and 2 miles North of its starting location (along the vector <1, 2>) and a hoverboard, defined similarly along the vector <3, 1>. As students work through this task and share solutions with classmates, they develop notation for linear combinations of vectors and connections between vector equations and systems of equations, providing support for representing the notion of linear combinations geometrically and algebraically.

The second task in the MCR sequence supports students' **referential activity** – activity in which students refer to and draw generalizations about physical and mental activity, often from the **situational activity** in the original task setting. In the second task, students are asked to determine whether there is any location where Old Man Gauss can hide from them if they were to use the same two modes of transportation from the previous problem. As students work on this task, they begin to develop the ability to conceptualize movement in the plane using combinations of vectors and also reason about the consequences of travel without actually calculating the results of linear combinations. This allows students to form conceptions of how vectors interact in linear combination without having to know the specific values comprising the vectors. The goal of the problem is to help students develop the notion of span in a two-dimensional setting before formalizing the concept with a definition. As with the first task, students are able to build arguments about the span of the given vectors and rely on both algebraic and geometric representations to support their arguments.

As students transition from the second task of MCR to the third, they have experience reasoning about linear combinations of vectors and systems of equations in terms of modes of transportation in two dimensions. In the third problem, students are asked to determine if, using three given vectors that represent modes of transportation in a three-dimensional world, they can take a journey that starts and ends at home (i.e., the origin). They are also given the restriction that the modes of transportation could only be used once for a fixed amount of time (represented by the scalars c_1, c_2, and c_3). The purpose of the problem is to provide an opportunity for students to develop geometric imagery for linear dependence and linear independence that can be leveraged through students' continued **referential activity** toward the development of the formal definitions of these concepts.

In the fourth task, students have the opportunity to engage in **general activity**, which involves students reasoning in ways that are independent of the original setting. In this task, students are asked to create their own sets of vectors for ten different conditions – two sets (one linearly independent and one linearly dependent) meeting each of the five criteria: two vectors in $\mathbb{R}^2$, three vectors in $\mathbb{R}^2$, two vectors in $\mathbb{R}^3$, three vectors in $\mathbb{R}^3$, and four vectors in $\mathbb{R}^3$. From their example generation, students create conjectures about properties of sets of vectors with respect to linear independence and linear dependence. This is **general activity** because students work with vectors without referring back explicitly to the MCR scenario as they explore properties of the linear in/dependence of sets of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$; furthermore, students often extend their conjectures to $\mathbb{R}^n$. Finally, students engage in **formal activity** as they use the definitions of span and linear independence in service of other arguments without having to re-unpack the definitions' meanings. This does not tend to occur during the MCR sequence but rather during the remainder of the semester as students work on tasks unrelated to the MCR sequence.

Effectiveness and Challenges of Inquiry-Oriented Instruction

Effectively implemented inquiry-oriented instructional approaches have been related to improved levels of conceptual understanding and equivalent levels of computational performance in areas ranging from K-12 mathematics, to undergraduate mathematics, physics, and chemistry (e.g., Cai, Wang N., Moyer, Wang, C., & Nie 2011; Deslauriers, Schelew, & Wieman, 2011; Kwon, Rasmussen, & Allen, 2005; Lewis & Lewis, 2005). To enact an RME curriculum, a classroom must engage students in inquiry into the mathematics of the problems posed. These classrooms are problem-based and student-centered, characteristics that overlap with other Inquiry Based Learning (IBL) and active learning classrooms (Laursen, Hassi, Kogan, & Weston, 2014). Consistent with others in the field (e.g., Kuster et al, 2017), in this work, we consider inquiry-oriented instruction to fall under the broader category of inquiry-based instruction. Research has shown that students who engage in cognitively demanding mathematical tasks have shown greater learning gains than those who do not (Stein & Lane, 1996). Furthermore, Stein and Lane (1996) found that those gains were greater in classrooms where students were encouraged to use multiple representations, multiple solution paths, and where multiple explanations were considered; in contrast, gains were lower in classrooms where the teacher demonstrated a process students could use to solve the task.

Implementation of the MCR task sequence described above is dependent on an inquiry-oriented classroom environment. Rasmussen and Kwon (2007) describe inquiry both as student inquiry into the mathematics through engagement in novel and challenging problems and instructor inquiry into students' mathematics to provide feedback to advance the mathematical agenda of the classroom. The MCR sequence is comprised of tasks that allow for multiple strategies and representations. Since the tasks are non-trivial, students are challenged with debating their answers and explaining their arguments. In addition, Tasks 2 and 3 each allow students to engage in mathematical activity that can be leveraged by the instructor to introduce formal definitions (span in Task 2, linear independence in Task 3). In both cases the instructor serves the role of broker between the classroom community and the mathematical community (Rasmussen, Zandieh & Wawro, 2009; Wenger, 1998) by taking student ideas and connecting them with the formal mathematical definitions. This brokering move of "interpreting between communities facilitates the students' sense of ownership of ideas and belief that mathematics is something that can be reinvented and figured out" (Zandieh, Wawro, & Rasmussen, 2017).

Game-Based Learning

Game Based Learning (GBL) is the use of digital games with educational objectives to significantly improve learning outcomes. Games are designed to be enjoyable and fun where students overcome challenges and goals (including educational goals) by gaining mastery of the rules within a constrained environment or setting (Dickey, 2005). Research in game-based learning has emphasized the importance of incorporating thoughtful learning theories into the design of games (Williams-Pierce, 2016; Gee, 2005; Gresalfi, 2015; Gresalfi & Barnes, 2016). Recently, there have been several GBL approaches that have been implemented in secondary and post-secondary classrooms (Sung & Gwo-Jen Hwang, 2013; Lester et al., 2014), most successfully when projects have used GBL in conjunction with an existing pedagogical approach (Salen, 2011; Shute, & Torres, 2012). Several learning and pedagogical approaches have been identified that align well with GBL (e.g., Barab, et al., 2012; Hamari, et al., 2016), and many projects approach learning from a constructivist perspective (e.g., Wilson, 1996; Kiili, 2005; Wu, et al., 2012). Curricula developed from constructivist perspectives typically engage students in activities in a problem-solving scenario so that students have opportunities to build on their understanding through reflective abstraction on their prior activity towards more advanced ways of thinking. We illustrate GBL with examples from *Rolly's Adventure* (RA), a videogame developed by the third author to support student learning about fractions.

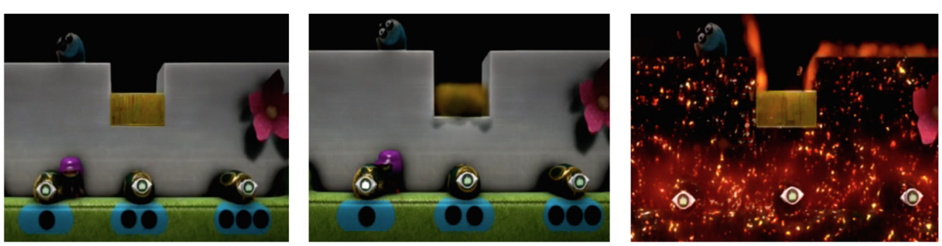

Figure 1: (a) The player (shown here in a purple helmet) enters the puzzle; (b) the player activates the first button; (c) the puzzle catches on fire.

RA begins with Rolly in the top left of the screen (see Figure 1). Rolly needs to roll past the obstacle (the gap) in the middle of the screen. The player's avatar is below Rolly in the purple hat. The player can choose from three options to press at the bottom of the screen. If the player chooses incorrectly the area explodes in fire and the golden bricks in the center show the result of the choice (see Figure 1c.)

In Figure 1, the player chose the single black circle and this did not change the size or shape of the golden brick. They then received feedback that their answer was incorrect (the fire that sends their avatar back to start over), and what the direct result of their action was (one black circle results in a single golden brick). Such instantaneous feedback and failure are considered crucial aspects of supporting learning during gameplay (e.g., Gee, 2005; Juul, 2009). If the player chooses the two black circles, the size of the bricks doubles to fill the space and Rolly (and thus the player) is able to move past the obstacle (see Figure 2), thus receiving positive feedback as to the accuracy of their choice.

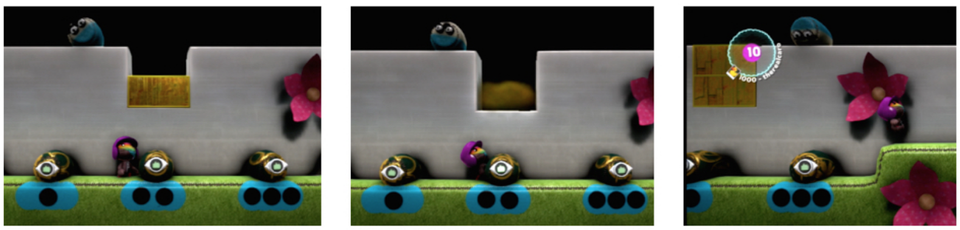

Figure 2: The player (a) activates the second button, (b) the bricks appear from a haze, and (c) successfully travels over the space now filled with bricks

As the player progresses through the challenges the brick or bricks in the obstacle will change in relationship to the space, and the way that the choices are indicated will also change. For example, the golden brick in Figures 1 and 2 represent one-half of the hole (the obstacle), and the next puzzle (not shown) has a block that represents one-fourth of the hole, following recommendations that halving a half is a natural next step in the learning of fractions (e.g., Empsen, 2002; Kieren, 1995; Smith, 2002).

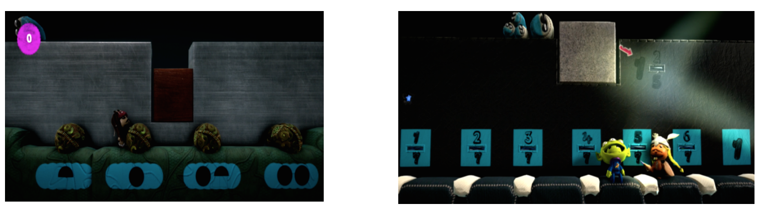

Figure 3: (a) is the fourth puzzle, where the brick is two-thirds of the hole; (b) is the fourteenth and final puzzle in RA, where the brick is one and two-fifths of the hole.

RA was designed specifically to begin with simpler puzzles and become more complex as players move through the trajectory, such that as players develop generalizations about the game, new puzzles emerge that continue to challenge and nuance these generalizations, so the player has a "pleasantly frustrating" experience (Gee, 2003). Accordingly, mathematical notation becomes introduced that supports the player in being more precise and accurate just as they begin to struggle, as a way of developing a sense of "intellectual need" (Harel, 2013) so that players find the notation immediately useful (following Gee, 2005). Figure 3 shows some examples of how the game becomes more complex. Note that the fourth puzzle (Figure 3a) has bricks that are not an integer multiple of the size of the hole. Correspondingly the player's options include whole and half circles. In the fourteenth puzzle (Figure 3b) the initial bricks are larger than the size of the hole and fraction notation is used to both label the relationship of the brick to the hole (one and two-fifths) and the different choices.

RA was designed specifically with GBL principles to support players in mathematizing their own gaming experience, and engaging in mathematical play (Williams-Pierce, 2017). In this fashion, RA served as a proxy for the role of the instructor in the brokering process (Rasmussen, Zandieh & Wawro, 2009; Wenger, 1998), in that the game required players to act as producers (Gee, 2003) in reinventing the mathematics underlying RA. In other words, an intentionally designed mathematics game can serve as a responsive digital context that mediates interactions between the player, the game, and the mathematical community. Ideally, a well-designed mathematics game uses the principles of failure and feedback to support players in experiencing a pleasantly frustrating and authentically mathematized world. In the following section, we focus more explicitly on how GBL, RME, and IO Instruction can be carefully blended in designs that evoke the best of each world.

Connecting GBL, RME, and IO Instruction - Blending Theoretical Worlds

The game design principles outlined above and illustrated with Rolly's Adventure align well with the nature of inquiry-oriented instruction using an RME-based curriculum. In Figure 4, we draw heavily on Gee's (2003) notion that good game design is good learning design to show parallels between principles of game design, RME curriculum design, and inquiry instruction and learning. Statements in the boxes of Figure 4 are all quotes or close paraphrases of various authors as indicated.

Looking across the rows in Figure 4 we see that both digital games and RME curricula place importance on the **structure of the task sequence**. The sequence should start with an activity in which students can immediately engage, but that has the potential to be generalized to a more sophisticated understanding that will help in solving more complex problems. We see this both in the increasing complexity of the tasks in Rolly's Adventure (RA) and in the magic carpet ride (MCR) tasks. In particular, student experiences graphically and imaginatively exploring the MCR scenario can be generalized to more formal notions of span and linear independence. As our project progresses, we can envision students being immersed in the MCR scenario through a digital game environment that allows for numerous episodes of growing complexity, from which student generalizations could emerge.

In considering the **nature of the tasks** we see that GBL, RME and IO/IBL all place emphasis on tasks that are novel and ill-structured allowing for a challenging but do-able problem-solving experience. The RA game (Williams-Pierce, in press) and the MCR tasks (Wawro et al., 2012) have both been empirically shown to be challenging, but manageable for students. A digital game based on the MCR sequence would share this novel approach. Through an iterative design process, tasks in the digital game can be created to be challenging but approachable for linear algebra students.

The **teacher's role** in inquiry classrooms is particularly important (Rasmussen & Kwon, 2007; Rasmussen et al., in press). Games can take on some of these roles. A well-designed game can intervene at desired junctures and provide real-time guidance or feedback based on the situation that the player is facing. A game can take on the role of the broker between the player (student) and the larger mathematics community. This brokering occurs both (1) through game play being consistent with the mathematical principles that the students are learning and (2) through students being gradually introduced to accepted mathematical notation and terminology.

	Theoretical Framing		
	GBL	RME	IO/IBL
Structure of task sequence	Good games confront players in the initial game levels with problems that are specifically designed to allow players to form good generalizations about what will work well later when they face more complex problems. [1]	Lessons should have experientially real starting points and engage in situational, referential, and general activity. [4]	----
Nature of the tasks	Good games operate at the outer and growing edge of a player's competence, remaining challenging, but do-able **...** **[**therefore**]** they are often also pleasantly frustrating, which is a very motivating state for human beings [1]	Challenging tasks, often situated in realistic situations, serve as the starting point for students' mathematical inquiry. [3]	IBL methods invite students to work out ill-structured but meaningful problems. [2] Students should solve novel problems [3]
Teachers' role	Good games give information "on demand" and "just in time," not out of the contexts of actual use or apart from people's purposes and goals… [1]	Teachers to build on students' thinking by posing new questions and tasks. [3]	Students present and discuss solutions; instructors guide and monitor this process. [2] Empower learners to see mathematics as a human activity. [3]
Students' role	Games allow players to be producers and not just consumers. [1]	Empower learners to see themselves as capable of reinventing mathematics [3]	Students construct, analyze, and critique mathematical arguments. Their ideas and explanations define and drive progress through the curriculum. [2]

Figure 4: Aligning three areas of our team's expertise that inform game design.
[1]Gee, 2003; [2]Laursen et al, 2014; [3]Rasmussen & Kwon, 2007; [4]Gravemeijer, 1999

Ultimately the first three categories are aimed at creating an optimal environment for student learning. The **students' roles** include producing ideas and explanations that allow for their guided reinvention of the mathematics. In RA players create increasingly nuanced generalizations as more complex situations are presented. Student creation of generalizations also occurs in the MCR sequence (Rasmussen, Wawro, & Zandieh, 2015). Our goals as we work toward creating a digital game based on the MCR sequence will be for players of this game to construct, analyze and critique mathematical arguments in the game scenario. For this to happen students need to both (1) experience the mathematical principles/structures through the feedback from gameplay and (2) reflect on their experiences and codify them in some way. In addition to having aspects of the game serve in the teacher role, the game may also need to have aspects that serve in the role of other students in the classroom with whom a student would collaborate in an IO or IBL setting (Ernst et al., 2017).

In conclusion, we believe that these overlapping aspects of GBL, RME and IO/IBL provide a solid starting point for creating a digital game based on the existing IOLA curriculum. As development progresses we will be able to explore affordances and constraints of the digital environment in comparison with the in-person IO classroom.

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