Developing Strategic Competence With Representations for Growth Modeling in Calculus

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Using inquiry based modules centered around growth modeling, we study the development of strategic competence and representational fluency in undergraduate calculus. Building on student experiences and using multiple representations with discrete and continuous methods, we discuss the emerging substantial and problematic practices with representational fluency, communication, and strategic competence for modeling growth.

Key words: Representational Fluency, Strategic Competence, Calculus, Differential Equations, Modeling

It has been suggested that mathematical modeling should be taught at every level of mathematics education (GAIMME, 2016), however successful modeling of realistic problems, like population dynamics, in STEM related fields requires students to achieve high levels of mathematical proficiency. The National Research Council defines mathematical proficiency as having five components, or interwoven strands: 1. conceptual understanding - comprehension of mathematical concepts, operations, and relations. 2. procedural fluency - skill in carrying out procedures flexibly, accurately, efficiently, and appropriately. 3. strategic competence - the ability to formulate, represent, and solve mathematical problems 4. adaptive reasoning - capacity for logical thought, reflection, explanation, and justification. 5. productive disposition - habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (NRC, 2001).

A crucial part of mathematical literacy, representational fluency refers to the ability to represent mathematical ideas with different representations, to translate these ideas across representations, to gain understanding about the underlying entities that are being represented, and to generalize across representations (Zbiek et al. 2007). It requires a metacognitive perspective requiring knowledge and synthesis beyond the representations themselves. This perspective was expressed by Sigel and Cocking as the ability to comprehend the equivalence of different modes of representation (Sigel and Cocking, 1977) after one can transfer information from one representation to another.

Despite the need for and benefits of representational fluency (e.g., Kaput, 1989), there is relatively little known about the calculus student's ability to solve problems when presented with different representations, or to translate ideas among different representations. Studies have reported that students have difficulties linking different representations and moving flexibly between representations (Even, 1998; Janvier, 1987). For example, researchers observed that calculus students were often comfortable with different results in different representations without realizing the inconsistency of the results (Ferrini-Mundy and Graham, 1993). Some researchers noted that students may link representations without an understanding of the deeper conceptual links between them (Greer & Harel, 1998). Beyond an equivalence perspective amongst representations, there is a need for a deeper look into how representational fluency translates to improved mathematical proficiency and strategic competence. More pointedly, little is known about how fluency among representations across discrete and continuous mathematics contribute to mathematical proficiency. Even (1998) highlighted that there is “not much known about the nature of the processes involved in working with different representations,” despite agreement among mathematics educators about their importance in learning mathematics.

Objectives and Research Questions

In this paper we discuss the collaborative action research of two mathematics faculty members with the goal of improving the practice of teaching calculus(Stinger, 2014). We infused collaboratively planned and purposefully designed inquiry based activities into a two semester freshman calculus sequence. Our activities were designed to provide opportunities for students to experience fluency with multiple representations from both a discrete and continuous perspective while investigating population growth modeling. Ultimately, we hope to further the development of both the strategic competence and the representational fluency in our students, and in doing so, to make the content of growth modeling more accessible for our student population. We both observed that this content is otherwise problematic with the traditional integration methods. Our main research questions are:

1. How do calculus students develop representational fluency when modeling population dynamics with an enriched instruction on discrete methods?
2. How do calculus students develop strategic competence when modeling population growth, specifically when they learn to connect complimentary discrete and continuous concepts, such as differential and difference equations?

Conceptual Framework for Representational Fluency in Growth Modeling in Calculus

Multiple external representations traditionally associated with mathematics have been outlined by many authors (Lesh, Post, and Behr, 1987; Kaput 1998; Kendal 2003); in this paper, we will refer to five different modes: Graphical, Algebraic, Verbal, Manipulative Models, and Real Life Scenarios (see Figure 1). Aligning with Kaput and Lesh, we take special care to incorporate real life scenarios and manipulative models, extending beyond just the big three representations.

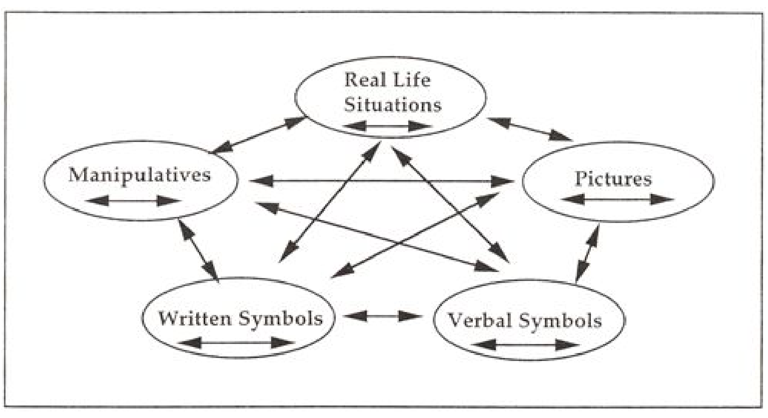

Figure 1. Lesh et al.’s model depicting five representational modes with Real Life Situations, Pictures/graphs, Written Symbols, Manipulatives/digital/concrete models, Verbal Symbols.

Each of the different representational modes affords the student different opportunities for mathematical insight. Advancements in technologies, the ease and availability of graphing calculators, and computer algebra systems now allow differentiation and integration to be easily calculated using numerical and graphical representations. Of course, these numerical and graphical solutions are primarily at a point or within an interval, rather than a global solution, as can be often found with the traditional analytical approach that relies on symbolic representations and algebraic manipulations.

In the context of calculus, and more specifically, growth modeling, students can demonstrate strategic competence by formulating modeling problems, by representing them with multiple representations, and by choosing flexibly among discrete or continuous methods to suit the demands of the mathematical content. Adaptive reasoning, on the other hand, refers to the capacity to think critically about the relationships among concepts and situations. Adaptive reasoning is the meta-cognitive leap to assess the fitness of the method and the adequacy of representations to provide the insight into problem in its realistic context.

We build on Rasmussen and Kwon's (2007) approach to inquiry based undergraduate mathematics by engaging our students in cognitively demanding tasks that prompt the exploration of important mathematical relationships and concepts, by orchestrating mathematical discussions in class and in small groups, by developing and testing conjectures, and by having students explain and justifying their thinking. Following an inquiry approach, we continually build upon, refine and expand our questions on population dynamics as we introduce new concepts in calculus. For example, we revisit population dynamics and present modeling opportunities at each step as we progress through major topics such as rates of change, anti-differentiation, and differential equations.

Methods and Setting

In Calculus I and II, we integrated both differential and difference equations as major components with instructors devoting approximately four weeks in each semester to these topics. Realistic scenarios were built around population growth, which was used as a cross-cutting theme that permeates across courses for the same group of students. The inquiry based modules that we infused into the calculus sequence emphasized discrete approaches to problems traditionally approached from a continuous perspective. The researchers collaboratively designed the modules used for this study since 2013. Our students were tasked with solving difficult problems in small groups by utilizing visual, analytical and verbal representations. Activities were purposely designed with the main goals of i. creating a more balanced approach to calculus with discrete and continuous methods; ii. Enhancing representational fluency; iii. Developing strategic competence.

We used multiple data sources, including analyzing student work, student reflections, and student discussions in an attempt to observe the student's representational fluency and strategic competence during the activities. The researchers also noted their observations and reflections on student behaviour and practices in follow-up discussions. Data was collected from students during the Fall and Spring Semesters of 2016 and 2017; in total, there were 23 students in Calculus I and 19 students in Calculus II.

An Integrated Calculus Instruction

As previously mentioned, the instructors spent approximately four weeks each semester engaging in inquiry based modeling activities focusing on discrete and continuous representations of population growth. For illustrative purposes, we offer a short description of two of the modeling activities we used, one from Calculus I and one from Calculus II. Aligning with recommendations from GAIMEE, we encourage our modeling problems to be approached in an open-ended manner to allow for the possibility of student conjecturing, exploration and investigation.

A Growth Modeling Activity in Calculus I

Students are presented with a modelling scenario involving the growth a fruit fly population, which was inspired by a similar problem in Thomas' Calculus(2014) that builds the idea of derivative from the rates of change of a logistic model given visually and numerically.

> *Imagine that one day a rotting apple in your kitchen has attracted some fruit flies. Suppose that on that day you count two fruit flies. You (unwisely) leave you home for 50 days, leaving the apple on your counter. When you return, the fly population has grown by 350 flies.*

We introduce alternative growth models before discussing the rate of change behavior for a logistic curve, not only with continuous but also with discrete methods. Our goal is to have students explore the given real world scenario and develop various models that can represent the growth of the population over the 50 day period, based on the assumptions that they formulate. The instructor ensures that the students represent their idea using multiple representation modes. In this case, most students are initially drawn to familiar continuous representations of linear growth, such as the (continuous) algebraic representation: $y = 7x+2$, the (continuous) graphical representation: a linear graph, and the verbal description of "a growth of 7 flies each day." If they choose this continuous approach, they are required to demonstrate their model using graphing technology (Geogebra or similar). The instructor asks questions which require manipulation of their model under different conditions (different initial population or growth rate, etc.). In our case, all students began the activity using this continuous approach.

Once they have successfully modeled linear growth with continuous methods, they are challenged to represent the growth using discrete methods. The students must now transfer ideas laterally among the same representation modes; for example, the represent growth algebraically with a difference equation: $P_{n+1} = P_n + 7$, graphically with a scatter plot, and they are asked to use a manipulative model such as Microsoft Excel to experiment with different parameters.

Students become aware of the limitations of the linear model and initiate investigations into other models, which we direct towards exponential and logistic growth. Once again, students must represent their ideas using algebraic equations, graphical images (see Table 1 below for more detail), and they must utilize manipulative models that can account for the changing of initial conditions. They are free to initiate either a discrete or continuous approach to their models, but through group collaboration, discussion, and reflection, all groups eventually see how these ideas can be modeled from both perspectives. Unifying questions relating the rates of change and the changes in the rates of change emphasize the complementary nature of the discrete and continuous approaches, and discussions involving the difficulties encountered by some approaches emphasize the importance of flexibility and strategic choice.

Continuing Growth Modeling Activity in Calculus II

In the second semester of Calculus, while studying first-order differential equations, students are presented with another modeling scenario involving a locally relevant invasive lionfish population:

> *Biologists have determined that a coral reef can safely sustain a population of 350 or fewer lionfish; however, once the population exceeds 350, irreversible damage will be done to the ecosystem.*

Once again aligning with GAIMEE recommendations, we allow students to formulate their own questions and ideas to investigate these scenarios. In this case, the instructors steered the students towards suggesting a harvesting strategy to keep the fish population below the threshold of 350. In previous modeling activities, students discovered a carrying capacity of 850 lionfish, and they proceed under that constraint. They make assumptions, such as an initial population, and the frequency of their harvesting expeditions, and proceed to answer questions like: *How many fish do we need to harvest if we send an expedition once every 6 months?* A continuous approach leads to representations like algebraic differential equations: $\frac{dy}{dx} =- 0.25y(1 - \frac{y}{850})$, continuous solution curves and slope fields, and manipulative models like slope field generators in GeoGebra. A discrete approach has students transfer between the difference equation: $P_{n+1} = 1.25P_n - \frac{0.25}{850} P_n^2$, and the graphical scatter plots made using Microsoft Excel (or similar), with which they can experiment with different parameter values. Ideas are summarized and presented to the class, so that discussion can ensue on the pros and cons of the different approaches.

Without including the graphical representations, we provide descriptions of the basic growth models introduced in modeling the population dynamics.

Table 1.
Summary for the Models for Population Dynamics

Underlying Math Models for Growth	*Contextual/ Verbal*	*Symbolic- Discrete*	*Symbolic Differential*
Linear	Constant Change	$\Delta P_n = P_n - P_{n-1} = k$	$\frac{dP}{dt} = k$
Exponential	Unbounded	$\Delta P_n = (a-1)P_{n-1}$	$\frac{dP}{dt} = (a-1)P$
Logistic	Limited Capacity	$\Delta P_n = mP_{n-1}(1 - \frac{P_{n-1}}{C})$	$\frac{dP}{dt} = mP(1 - \frac{P}{C})$

Observations and Results

Our observations suggest that the enhanced treatment of growth modeling with a balanced focus on discrete and continuous methods can improve the development of representational fluency and strategic competence in participants. On several occasions, we observed what we perceived as higher than usual student growth in the ability to transfer ideas across traditional representational paths, such as from continuous equations to continuous graphical representations. For instance, the majority of students (72%) that were unable to correctly connect a differential equation to its direction field prior to our modeling activities were able to successfully do so afterwards. The assessment question used in this case is seen in Figure 2.

Which of the following equations is the differential equation whose slope field is shown below?

a) y'=3-y-x
b) y'=3+y-x
c) y'=3-y+x
d) y'=3+y+x
e) y'=3-yx

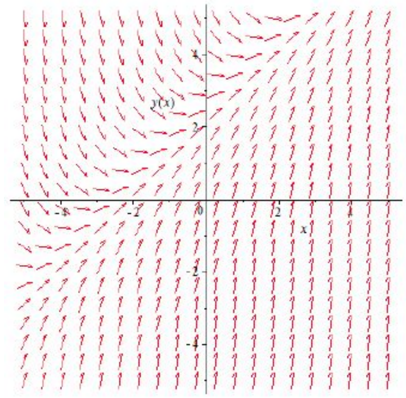

Figure 2. Slope field assessment task.

In addition, we also observed that upon the completion of our course, students were demonstrating an enhanced flexibly in choosing among discrete or continuous methods that best suited the problem at-hand. In our initial assessments, students would largely prefer continuous approaches, regardless of the comparative difficulty of discrete approaches. For example, in our unit on arc length, students were tasked with the well-known problem of finding the length of the Golden Gate Bridge, which is modeled with the equation $y = \frac{x^2}{8820} - \frac{10x}{21} + 500$, but only to within 10 feet of accuracy. We observed that 4 of 5 groups pursued an exact solution via the continuous integration formula, whereas the remaining group solved the problem using a line segment approximation. We note that students had practiced such approximations recently. In fact, all four groups were unable to solve to the continuous integral, and resigned the problem rather than switch approaches. After completing our modeling exercises, students attempted the following question:

> *The population of lionfish in a water column above a coral reef near Buck Island is given by* $\frac{dy}{dx} =- 0.15y(1 - 0.06375y + \frac{y^2}{12800})$ *where y is the population in lionfish and x is in years. Biologists determine that the reef can safely sustain a population of 350 or fewer lionfish, but once the population exceeds 350 irreversible damage will be done to the ecosystem. A diving survey team estimates a current population of 180 lionfish. After approximately how many months will the population equal 350?*

This time, the majority of the groups (4 out of 5) used a discrete approach (Euler's method) for their initial strategy; whereas the remaining group began with a continuous approach, but were able to switch the the discrete method after some initial failure. We further make note of our observation of what we perceived to be better than expected results in the student's ability to formulate and solve modeling problems. Groups engaged in the harvesting exercises demonstrated more mathematical autonomy and independence in completing their assigned tasks. It was clear that student's strategic competence became amply evident in growth modeling tasks when the instruction allows student experimentation with manipulatives, such as the dynamic spreadsheets that blend the numerical or graphical representations. Our students performance exceeded our expectations with their problem formulating skills, their critical thinking in the creation of their models, and their suggestions for harvesting schemes for population control. They also seemed to become more productive and reflective after strategic choices of visual representations, such as flow diagrams, substantially empowered them towards a dynamic sense of the global behavior of solution curves under different initial conditions.

By the culmination of the activity sequence, we observed students development in both the cognitive and content related skills in calculus, such as representational fluency and building connections between discrete and continuous methods in modeling growth. Our final remark is that the additional fluency involving the discrete representational forms emerged in a critical capacity as providing certain students access to deeper mathematical ideas that were inaccessible to them from a continuous standpoint. We observed that several students had difficulty solving growth problems analytically, in particular, when modeling logistic growth, however, they ultimately overcame their earlier problems producing solution curves algebraically when asked to use traditional integration techniques in calculus. Most of our students who struggle with difficult concepts in topics like differential equations were more able to experience success with this approach, as exemplified in the harvesting activity outlined above. Their exercised ability to use spreadsheets allowed even the weakest students to see the impacts of harvesting at set time periods clearly.

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