Collective Argumentation Regarding Integration of Complex Functions Within Three Worlds of Mathematics

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Although undergraduate complex variables courses often do not emphasize formal proofs, many widely-used integration theorems contain nuanced hypotheses. Accordingly, students invoking such theorems must verify and attend to these hypotheses via a blend of symbolic, embodied, and formal reasoning. This report explicates a study exploring student pairs' collective argumentation about integration of complex functions, with emphasis placed on students' attention to hypotheses of integration theorems. Data consisted of task-based, semistructured interviews with pairs of undergraduates, as well as classroom observations. Findings indicate that participants' explicit qualifiers and challenges to each other's assertions catalyzed new arguments allowing students to reach consensus or verify conjectures. Although participants occasionally conflated certain formal hypotheses, their arguments married traditional integral symbolism with dynamic gestures and clever embodied diagrams. Participants also took care to avoid invoking attributes of real numbers that no longer apply to the complex setting. Teaching and research implications are discussed as well.

Keywords: Collective Argumentation, Complex Variables, Integration, Reasoning

Introduction and Literature Review

Although the discipline of mathematics often rests on generalizing results from one domain to another, at times "mathematical thinking may involve a particular manner of working that is supportive in one context but becomes problematic in another" (Tall, 2013, p. xv). Such considerations can arise when studying the teaching and learning of complex analysis. For example, Danenhower (2000) discovered a theme of "thinking real, doing complex" (p. 101) wherein participants invoked attributes of real numbers that do not necessarily apply in the complex context. Troup (2015) additionally evidenced this phenomenon when undergraduates reasoned about complex differentiation. Within the setting of real-valued functions, the literature abounds with examples of students' struggles with integration (Grundmeier, Hansen, & Sousa, 2006; Judson & Nishimori, 2005; Mahir, 2009; Orton, 1983; Palmiter, 1991; Rasslan & Tall, 2002). However, most of these studies showcased the *product* of students' deficiencies and misconceptions rather than the *process* of students' reasoning. Accordingly, although students might sometimes draw incorrect conclusions regarding integration, their process of reasoning may healthily appeal to intuition or past experiences. When cultivated properly, such connections between experientially-based intuition and formal mathematics could benefit students' reasoning in courses such as complex variables (Soto-Johnson, Hancock, & Oehrtman, 2016).

Furthermore, according to Wawro (2015), by researching students' successful reasoning about undergraduate mathematics topics, we can document "what deep understanding and complex justifications are possible for students as they engage in mathematics" (p. 355). The subject of complex variables is particularly amenable to such an investigation, as the content in this course often lies between symbolic calculation and formal proof. Specifically, students that integrate complex functions tend to apply powerful theorems that rely on idiosyncratic hypotheses and draw on notions from real analysis and/or topology. Though formal proof is typically not emphasized in undergraduate courses in complex variables (Committee on the

Undergraduate Program in Mathematics, 2015), the application of such theorems requires that students at least recognize when these hypotheses apply. As such, students may invoke a blend of intuition, visualization, symbolic manipulation, and formal deduction when integrating complex functions. Accordingly, integration of complex functions lends itself to eliciting the rich student justifications called for by Wawro. Complex integration also has numerous practical applications for students, such as computing flux, potential, or certain real-valued integrals.

Despite these practical and theoretical assets, no existing educational research examines undergraduates' reasoning about integration of complex functions. In particular, researchers have not yet documented how students reason algebraically, geometrically, and formally when integrating such functions. This study served to ameliorate this gap in the literature and to inform the teaching and learning of complex variables by analyzing undergraduates' multifaceted argumentation about integration of complex functions. Using Tall's (2013) Three Worlds of Mathematics framework, my research sought to answer the following guiding questions:

1. How do pairs of undergraduate students attend to the idiosyncratic assumptions present in integration theorems, when evaluating specific integrals?
2. How do pairs of undergraduate students invoke the embodied, symbolic, and formal worlds during collective argumentation regarding integration of complex functions?

In this study, argumentation is defined according to Toulmin's (2003) model consisting of six components: data, warrant, backing, qualifier, rebuttal, and claim. Given that my study considers how pairs of students reason about integration tasks, it is additionally important that I consider how each individual contributes to an argument. Accordingly, I adopt Krummheuer's (1995) notion of *collective argumentation* in which multiple participants construct arguments through emergent social interaction. These interactions involve four speaker roles (author, relayer, ghostee, and spokesman), classified according to how syntactically and/or semantically responsible an individual is for the content of his or her statement. Readers unfamiliar with these speaker roles may consult Krummheuer (1995) or Levinson (1988) for more information.

The existing mathematics education literature implementing Toulmin's model manifests itself in several contexts. In the in-class setting, some researchers (Krummheuer, 1995; Krummheuer, 2007; Rasmussen et al., 2004; Stephan & Rasmussen, 2002) used a reduced Toulmin model omitting the qualifier and rebuttal, and rarely evidenced explicit backing. However, when more formal arguments such as proofs were concerned, researchers (Alcock & Weber, 2005; Inglis, Mejia-Ramos, & Simpson, 2007; Simpson, 2015) argued for the use of the full Toulmin model. These researchers also highlighted that simply reading the finished product of a purported proof is inherently difficult because backing and warrants are often implicit and cannot be elicited through real-time discourse with the proof author. Thus, my investigation into undergraduates' nuanced argumentation about integration of complex functions incorporated the full Toulmin model as well as opportunities for clarification in an interview setting.

Theoretical Perspective

This work is theoretically oriented by Tall's (2013) Three Worlds of Mathematics as a lens through which to analyze undergraduates' reasoning pertaining to integration of complex functions. Tall's perspective situates mathematical knowledge within three distinct but interrelated forms of thought: *conceptual-embodied*, *operational-symbolic,* and *axiomatic-formal.* Conceptual embodiment begins with the study of objects and their properties, and can incorporate mental visualization. Operational symbolism grows out of actions on objects and can be symbolized flexibly as *procepts*, symbols operating dually as process and concept (Tall, 2008). The world of *axiomatic formalism* attends to "formal knowledge in axiomatic systems specified by set-theoretic definition, whose properties are deduced by mathematical proof" (p. 17). These three worlds can also combine to form, for example, *embodied-symbolic* or *formal-embodied* reasoning. As mentioned earlier, our prior experiences with mathematics can either support or clash with new and abstracted mathematical notions. Tall refers to the mental schemas predicated on these prior experiences as *met-befores*. He also argues that mathematical growth is afforded by three innate *set-befores* of recognition, repetition, and language. These set-befores foster categorization, encapsulation, and definition in order to compress knowledge into *crystalline* structures, which house various equivalent formulations of a mathematical object and can be unpacked in various worlds.

Moreover, "each world develops its own 'warrants for truth'" (Tall, 2004, p. 287). In the embodied and symbolic worlds (respectively), truth derives from what is *seen* to be true by the learner visually, and from calculation. Yet in the formal world, a statement is either assumed as an axiom, or can be proven from axioms. Hence, Tall's three-world perspective can complement the Toulmin analysis of a mathematical argument by adding specificity with regard to the types of backing and warrants used. As such, I classify participants' Toulmin components as embodied, symbolic, formal, or various mixtures of these, as viewed through Tall's three-world lens. Consequently, I define *reasoning* as mathematical argumentation within one or more of the three worlds. I also garner specificity by adopting Simpson's (2015) three classifications of backing. Specifically, *backing for the warrant's validity* explains why a warrant applies to a given argument. A second type serves to "highlight the logical field in which the warrants are acceptable," which Simpson characterized as *backing for the warrant's field* (p. 12). The third type, *backing for the warrant's correctness*, demonstrates that a given warrant is actually correct.

Methods

In order to rigorously address my research questions, I enlisted the help of two pairs of undergraduate students to partake in a videotaped, semistructured (Merriam, 2009), task-based interview comprised of two 90-minute portions and 13 tasks. Participants were selected from undergraduate students at a military academy in the United States, enrolled in a complex variables course during the spring 2015 semester. My first pair of participants consisted of Sean and Riley. Sean was a fourth-year physics and mathematics major and Riley was a second-year applied mathematics major with a cyberwarfare concentration. The second pair consisted of Dan, a third-year mathematics major, and Frank, a second-year applied mathematics major with an aero concentration. All participants' names listed here are pseudonyms. A sample analysis of interview data is detailed in the next section.

To obtain a rich understanding of the context in which these participants learned about integration of complex functions, I also observed and videotaped six class sessions at participants' undergraduate institution. These observations and ensuing field notes allowed me to document what mathematical content was introduced and emphasized during the integration unit in the complex variables course. They also allowed me to discern the nature of mathematical argumentation that was deemed appropriate for the complex variables course. For the sake of brevity, I restrict the presentation of results here to my interview findings. I also note here that I read tasks aloud verbally during the interviews so as not to overtly suggest any particular representation or world to participants.

Results and Discussion

Due to my definition of reasoning in the context of this study as collective argumentation within one or more of Tall's (2013) three worlds, I format my results within each task according to argument. Included in my account of each collective argument are: pertinent excerpts of the participants' interview transcript; a Toulmin (2003) diagram summarizing the argument; and figures illustrating participants' gestures or inscriptions, often for the purpose of documenting embodied reasoning. Because of page constraints, this report showcases select results from Riley and Sean's interview. In particular, I present analysis of Riley and Sean's response to one task, in which participants evaluated the integral $\int_L \frac{1}{z}\,dz$, where L denotes the unit circle $|z| = 1$ traversed counterclockwise. Afterwards, I allude to general findings from both pairs' interviews, and discuss various implications of my work.

Sample Task Analysis

In illuminating Riley and Sean's reasoning about the task, I reference line numbers from their transcript excerpts and refer to various components of the Toulmin diagrams I constructed based on my interpretation of their responses. I also convey individual participants' speaker roles germane to each Toulmin component in the collective argument. Throughout the transcript pieces and Toulmin diagrams presented in this section, 'Int.' signals statements that I said aloud as the interviewer, while 'R' and 'S' stand for Riley and Sean, respectively. Bracketed phrases represent non-verbal events such as gestures or written inscriptions produced by the participants. In the Toulmin diagrams, italicized statements represent participants' exact verbiage from the transcript, while non-italicized statements more succinctly summarize participants' reasoning or deduce implicit Toulmin components based on their verbiage, gestures, and inscriptions, or lack thereof. Horizontal and vertical lines show how argumentation components are linked within a collective argument or subargument. Following the format of Wawro (2015), I represent shifts in the Toulmin categorization from one type of component to another (such as claim to data) in the figures by a diagonal line.

As I read the task aloud, Sean symbolically relayed the data comprised of the integral $\int_L \frac{1}{z}\,dz$ and the path $|z|= 1$ (line 4). He also authored an embodied datum by drawing the circular path on an Argand plane (see Fig. 1). As spokesman, Sean then symbolically rewrote L as $C_1^+(0)$, and I acknowledged this alternate symbolism from their class (lines 4-5). Riley agreed, but Sean made sure to document that this was the professor's notation, as if indicating that he did not hold any agency when using it (lines 6-8).

Sean proceeded as spokesman, indicating that they could apply an antiderivative, as in the last task (lines 9-10). He also qualified this suggestion with the phrase, "I think I'm pretty sure that…" (line 9). However, Riley challenged Sean as she authored a warrant: "There's no branch we can choose […] so that [the integrand] is going to be analytic over the entire path" (lines 11-12). Invoking embodied reasoning, Riley also revised Sean's initial diagram of the circular path to include a positive orientation (see Fig. 1).

Int: Ok so more integration stuff. And I think given what we just discussed, I think this one shouldn't be too bad for you. So how would you find the integral of $\frac{1}{z}\,dz$, and this is over the path L, if L represents the unit circle $|z| = 1$, traversed counter-clockwise?
S: [Writes $\int_L \frac{1}{z}\,dz$, |z|= 1, and draws the path on an Argand plane. Changes notation of L to $C_1^+(0)$]
Int: Yeah, and I realize you guys have your own notation for that too, so—
R: Yeah.
S: Dr. X's notation. I'll document that [laughs].
Int: Right [laughs].
S: Ok so kind of like we did last time, I think I'm pretty sure that— I normally call it using the antiderivative, but—
R: [Draws in orientation on path] There's no branch we can choose, right, so that it's going to be analytic over the entire path?
S: Yeah, so I'd just say, $z = e^{i\theta}$ [writes this]. Therefore $z'(\theta) = ie^{i\theta}$ [writes this]. So we say the integral from 0 to 2π, because theta is of course from these values [writes $0 \leq \theta \leq 2\pi$], 1 over $e^{i\theta}$, times $ie^{i\theta}\,d\theta$ [writes $\int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta}\,d\theta$]. Which then you're going to get $i\int_0^{2\pi} d\theta$ [writes this], $2\pi i$ [writes $= 2\pi i$], which gives the well-known result of integral of 1 over z minus a pole [points to $\int_L \frac{1}{z}\,dz$ inscription], a circle centered around it [points to diagram of path], gives you $2\pi i$ [points to $2\pi i$ inscription].

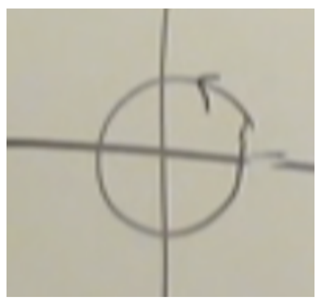

Figure 1. Sean's diagram for the path L and Riley's counterclockwise orientation in Task 6.

Sean conceded, and used their warrant to author an alternate approach implementing parametrization. Specifically, he first used embodied-symbolic reasoning to conclude that $z = e^{i\theta}$ is a parametrization of their path (line 13). Using this now as a datum, he further concluded that $z'(\theta) = ie^{i\theta}$, evidencing symbolic reasoning (line 13). As spokesman, Sean implemented embodied-symbolic reasoning to re-write the original integral, incorporating this new parametrization. The embodied aspect of this rewriting came from the decision to allow theta to vary from 0 to 2π, a decision qualified by the phrase, "theta is of course from these values" (lines 13-15). Sean symbolically simplified this integral to obtain $i\int_0^{2\pi} d\theta$, and claimed that they obtained the "well-known result" of $2\pi i$ (lines 16-18). This sole argument for Task 6 is summarized in Figure 2.

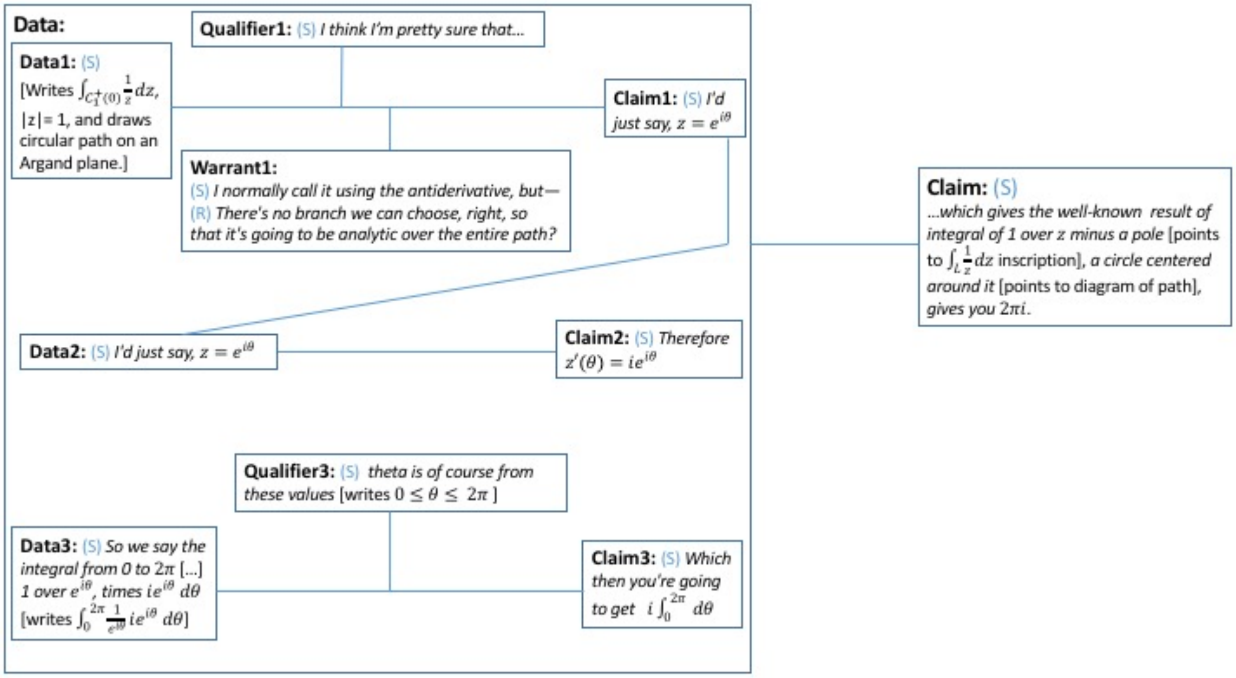

Figure 2. Toulmin diagram for Riley and Sean, Task 6.

General Findings and Implications

With the above sample analysis in mind, I now elucidate some general findings that address my aforementioned research questions. I also discuss teaching and research implications associated with these results. Recall that my first research question regarded how undergraduate student pairs attended to the assumptions pertaining to integration theorems. In the present study, neither pair of participants appeared confident nor certain about the assumptions needed for employing certain tools, approaches, or theorems. For instance, Riley and Sean repeatedly questioned themselves in a previous task about whether the integrand function needs to be differentiable in order to employ parametrization. By explicitly qualifying such arguments, and in conjunction with my follow-up questioning, they eventually reached a consensus that the function only needs to be continuous. However, because they did not spend significant time in their course carefully justifying continuity arguments, the students exhibited substantial difficulty justifying why given functions, such as $\bar{z}$, are continuous or not. In particular, they pursued limit calculations to try to show this function was not continuous, but muddled their symbolic limit inscriptions.

Although Dan and Frank exhibited more confidence and decisiveness when deciding a function's continuity, they faltered a bit when justifying their application of Cauchy's Integral Formula in the above task. In particular, when Dan claimed they could produce a simply-connected domain containing the path L, Frank questioned the existence of such a domain, and his attempt at drawing one resulted in a domain that was not simply-connected. However, as with the above example, Dan and Frank's eventual consensus resulted from an explicit modal qualifier. The importance of such explicit qualifiers across the interviews was that they often led to follow-up arguments wherein the participants discussed assumptions in greater detail, including their applicability to the integral at hand. As such, my findings corroborate previous researchers' contention that one should consider the full Toulmin (2003) model when analyzing undergraduate level mathematical arguments.

My second research question inquired about the nature of students' invocation of Tall's (2013) three worlds during collective argumentation about complex integration. Quite unsurprisingly, my participants' formal reasoning dealt primarily with Cauchy's Integral Formula, the Cauchy-Goursat Theorem, the Cauchy-Riemann equations, and related results when evaluating specific integrals. However, more illuminating were the ways in which participants instantiated formal-symbolic, formal-embodied, or embodied-symbolic reasoning to justify the implementation of such theorems. For instance, Riley (and eventually Sean) explicitly instantiated embodied-symbolic reasoning by drawing arrows on the whiteboard between symbolic inscriptions and embodied paths of integration which were drawn on the board. Participants also expressed a symbolic answer next to a particular embodied path of integration by writing "$= 0$" next to a diagram of a closed path containing no singularities, for example. When discussing limits and path-independence, all four participants produced symbolic limit inscriptions, but also conveyed corresponding dynamic gestures embodying their chosen paths of approach. In one task, Riley and Sean demonstrated a purely embodied method for integrating the conjugate $\bar{z}$. The pair plotted tangent vectors along the circular path of integration and conjugates resulting from reflection transformations, and Riley and Sean also enacted visual vector addition.

Accordingly, the manners in which students intertwined embodied reasoning with symbolic and formal reasoning highlight the importance of visualization and geometry in the study of complex integration. Although complex variables courses tend to focus on symbolic computations and applications involving integration, the above examples point to an important consideration for teaching such a course. Specifically, they suggest that instructors might want to more explicitly highlight how the symbolism that abounds during the integration unit of a complex variables course can intertwine with the embodied and formal worlds. For instance, after providing a formal definition for a simply-connected domain or a simple curve, students could benefit from drawing numerous examples and counterexamples with one another. At times, my participants conflated some of these formal requirements, suggesting that additional care should be taken to produce examples that satisfy one requirement but not another. Despite participants' occasional struggles with formal hypotheses, both pairs were cognizant of the *thinking real, doing complex* (Danenhower, 2000) phenomenon, and avoided inappropriate applications of it. For instance, all participants voiced concerns such as "I'm tempted to think of this in terms of real numbers, but I know the analogy doesn't work" at various times during the interviews.

Finally, my study complements and extends the mathematics education literature regarding students' mathematical argumentation, particularly regarding how Toulmin's (2003) model is adopted to the context of *collective* argumentation. Specifically, not only did my participants' explicit qualifiers catalyze new arguments, but follow-up arguments also ensued when individuals challenged each other's assertions. According to Krummheuer (2007), individuals participate in collective argumentation in two ways: (1) the production of statements categorized according to Toulmin's model, and (2) an individual's speaker role (author, relayer, etc.). Notice that both of these forms of participation primarily serve to either introduce new ideas or support/re-voice existing ideas. However, they do not account for disagreement between parties or changing one's own mind following internal reflection. Accordingly, I contend that a third type of participation can drive collective argumentation, namely *challenging*.

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