Students' Usage of Visual Imagery to Reason about the Divergence, Integral, Direct Comparison, Limit Comparison, Ratio, and Root Convergence Tests

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This study was motivated by practical issues we have encountered as second-semester calculus instructors, where students struggle to make sense of the various series convergence tests, including the divergence, integral, direct comparison, limit comparison, ratio, and root tests. To begin an exploration of how students might reason about these tests, we examined the visual imagery used by students when asked to describe what these tests are and why they provide the conclusions they do. It appeared that each test had certain types of visual imagery associated with it, which were at times productive and at times a hindrance. We describe how the visual imagery used by students seemed to impact their reasoning about the convergence tests.

Key words: calculus, sequences, series, convergence tests, visual reasoning

This study was motivated by practical issues we have encountered as second-semester calculus instructors, where students work with the concepts of sequences (a_n), series ($\sum_{n=1}^{\infty} a_n$), and the notion of convergence. Students are typically supposed to learn several convergence tests that can be used to determine whether a given series will converge or diverge. Students struggle with these, both in terms of calculation and in terms of reasoning about why these convergence tests work (i.e. why the results are valid). In appealing to the research literature for insight into this topic, we found that while studies have examined student understanding and reasoning about sequences and series (e.g., Alcock & Simpson, 2004; Mamona-Downs, 2001; Martinez-Planell & Gonzales, 2012; McDonald, Mathews, & Strobel, 2000; Tall, 1992; Tall & Vinner, 1981), there is very little work done about how students reason about *convergence tests* specifically.

González-Martín, Nardi, and Biza (2011) noted that visual representations of convergence were mostly limited to depictions of the integral test. Earls and Demeke (2016) have found that students made many types of mistakes or errors when testing convergence, such as using an inappropriate convergence test or failing to check whether a series meets the criteria for a given convergence test. Earls (2017) also found that students may confuse sequence convergence with series convergence while performing these tests, sometimes imagining them as interchangeable. While these few studies have provided an initial foray into the topic of convergence tests, we feel there is much work to be done in this area. This study is meant to contribute by investigating the question: How do students reason about the convergence tests presented in second-semester calculus? Additional specificity about "reasoning" is given in the "Framework" section.

Recap of the Convergence Tests

In our study, we examined the divergence, integral, direct comparison, limit comparison, ratio, and root tests. For our purposes, the "p-test" is considered a special case of the integral test and what we call "direct comparison" is often simply called the "comparison" test. The divergence test states that if $\lim_{n\to\infty} a_n \neq 0$, or does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges. The integral test states that if $f(x)$ is continuous, positive, and decreasing on $[1,\infty)$, and $a_n = f(n)$, then the series converges if and only if $\int_1^{\infty} f(x)dx$ converges. The direct comparison test begins with the assumption that $0 \leq a_n \leq b_n$ for all n. Then, if $\sum_{n=1}^{\infty} b_n$ is convergent, so is $\sum_{n=1}^{\infty} a_n$. If $\sum_{n=1}^{\infty} a_n$ is divergent, then so is $\sum_{n=1}^{\infty} b_n$. The limit comparison states that for positive sequences

a_n and b_n, if $\lim_{n\to\infty}(a_n/b_n) = c$ where $0 < c < \infty$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge. The ratio test begins with the assumption that $\lim_{n\to\infty}|a_{n+1}/a_n| = L$. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent. If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent. If $L = 1$, the test is inconclusive. The root test begins with the assumption that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent. If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent. If $L = 1$, the test is inconclusive.

Framework: Visual Imagery

For this study, we narrowed our scope on reasoning by focusing on reasoning based on *visual* imagery. We chose a visual imagery perspective because it has been well-studied and advocated for in mathematics education research. Also, it has been applied specifically to sequences and series (Alcock & Simpson, 2004), and there is a small amount of information known about how textbooks use visual imagery for the integral test (Martinez-Planell & Gonzales, 2012). For our framework, we began with Presmeg's (1986, 2006) five visual imagery categories, shown in the first five lines of Table 1. During analysis, we also remained open to the possibility of additional visual imagery types being added to this framework. In fact, we identified two useful additional categories (the last two lines in Table 1). First, many students used the *visual appearance* of the symbols and expressions in reasoning. Second, students often used their hands to *spatially locate* conceptual objects, like a sequence or a series, in the physical space in front of them, similar to the use of a "signer's box" in sign language. These two categories were added to our framework.

Table 1. Definitions and operationalization of the visual imagery categories we used in our study.

Imagery	**Definition**	**Operationalization: The student…**
Concrete	Static image or picture in the mind	…produces a non-moving/changing image, whether with pencil/pen/fingers or verbally described.
Pattern	Imagined relationship stripped of concrete details	… quickly recites a specific structure or pattern pertaining to a convergence test.
Memory of Formula	Image recall of a literal formula or expression	…writes, gestures, or verbalizes a generic symbolic template associated with a convergence test.
Kinesthetic	Imagery inherently using physical movement	…uses bodily movement as an inherent part of the initial image (not just for communication purposes).
Dynamic	A static image that is then moved or transformed	…begins with a static image and then describes it as transforming (including if gestures are used).
Symbol Appearance	Visual look of a symbol or symbolic expression	…uses the way a symbol or expression looks to make a conclusion, comparison, or connection.
Spatial Location	Physically locating a conceptual object in space	…uses gestures to "place" an object in physical space around them, or to references those objects.

Methods

We used semi-structured interviews with nine second-semester calculus students (labelled A–I) who had recently learned about series convergence tests in their calculus course. The students came from the same "large-lecture" calculus course (~250 students), which was taught in a fairly traditional manner. However, the instructor did attempt to incorporate discussions about what series convergence meant, typically centered on sequences terms becoming small "fast enough" for the series to converge. The students were selected based on their performance to three series convergence questions, with four correct on all three questions and five correct on two questions.

In the interviews, the students were asked to discuss the six convergence tests. For each test, the students were asked (a) to describe what that test is, (b) to explain why that test works for assessing convergence, and (c) to explain why the conclusions of each test are justified. The first part gave us information on the student's knowledge of each test's contents, and the second and third parts allowed us to examine the visual imagery and visual reasoning used by the students.

The analysis consisted of four phases. In the first phase, we reviewed the interviews to look for additional possible visual imagery categories, which produced the two described previously. In the second phase, we applied our visual imagery framework (Table 1) to the data, coding every instance in each interview that fit within any of those categories. Our coding was then independently checked by a separate research assistant. Any changes made by that assistant were reviewed again by us, to make final decisions. For each student, counts were made for how many instances of each type of imagery occurred during their discussion of the individual convergence tests. Aggregate counts were also tabulated across all nine students. In the third phase, we noted trends of how each type of visual imagery was typically used across the students. This was done by grouping together all instances of one type of imagery and looking for commonalities. Then, in the fourth phase, we examined how the different types of imagery influenced how students reasoned about each type of convergence test. This was done by looking at whether certain types of imagery proved helpful, or not, for making sense of the each convergence test.

Results

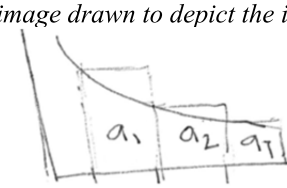
We organize the results as follows. First, we describe, generally, the trends we saw for how each type of visual imagery was used across the students. Then, we provide summary frequencies for each kind of visual imagery used for each convergence test. Last, we explain how these types of visual imagery affected how the students reasoned about each convergence test.

How Students Generally Used Each Visual Imagery Type

To acquaint the reader with the overall trends in student imagery use, we start our results by first describing the common ways that each type of visual imagery was present in the data set.

Concrete imagery. Students frequently used concrete imagery. For the integral test, they drew the "typical" textbook image (Figure 1) of a continuous function passing through the "dots" representing the sequence, with rectangles having heights equal to the sequence values.

Figure 1. Typical concrete image drawn to depict the integral test (from Student C).



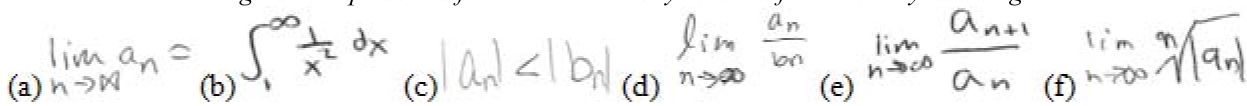
In fact, students drew many decreasing "functions," either on paper or in the air. Students also used concrete imagery to imagine sequences as an ordered list of numbers a_1, a_2, a_3,... . This was evidenced by gestures where students would "point" to the imagined successive numbers as they described a sequence. Concrete imagery was also frequently used by students to imagine "sizes" of numbers or sequence terms. For example, students used their index finger and thumb to make small or large "size" gestures, or they used the distance between their hands to show size.

Pattern imagery. Overall, there was less evidence for pattern imagery. The main instances of this imagery were students quickly reciting the pattern of a convergence test's results. For example, when describing the ratio or root test, many students quickly recited the $L > 1$, $L = 1$, and $L < 1$ cases. Students seemed to have this pattern laid out visually in their minds, as they would sometimes point "up" when referring to the $L > 1$ case and "down" when referring to the L

< 1 case. Pattern imagery was also used, but much less so, for limit comparison, in quickly stating that $0 < c < \infty$ implied one thing and that 0 or ∞ implied the opposite.

Memory of formulae imagery. Each convergence test had an expression type recalled from memory by the students directly in its symbolic form (see Figure 2). These expressions seemed invoked as a single visual unit when the students initially discussed a given test. Note that for the integral test, students typically *verbally* stated that the integrand was the function obtained by replacing "n" in the sequence with "x," but usually wrote an example rather than a generic $f(x)$. For some expressions, there were variations in how the students imagined it, such as the expression a_{n+1}/a_n with no "$\lim_{n\to\infty}$" on it, or some expressions having absolute values or not.

Figure 2. Expressions for each test directly recalled from memory as a single unit



Kinesthetic imagery. This imagery was the most extensively used by the students, largely because of the kinesthetic nature of how the students seemed to think about convergence and divergence. This is likely related to the instructor's discussions of sequence terms becoming small "fast enough." Students frequently gestured off to the right when talking about these concepts, sometimes "downward" for convergence and "upward" for divergence (though not always). Another common place this type of imagery was used was when students thought of the series as summing up the terms of the sequence. The students sometimes made gestures like "collecting" terms together, or "grabbing" them one by one, implying *action* during summation.

Dynamic imagery. Dynamic imagery mostly showed up when students began with a *symbolic expression* and then imagined it transforming in some way. We highlight that this is different from symbolic manipulation, because it was the *imagery* associated with the expression itself that changed. For example, Student G, in describing how the root test is convenient for cancelling off n-th powers, stated, "If you look at just what's *inside* [student emphasis] the n-th power, you'll kind of have a better idea of what's going on." This statement was accompanied by a gesture where he put his rounded his hands next to each other and then shrunk the space between his hands as though "zooming." Instances like this one suggest that the students were implicitly imagining some transformation to the symbolic expression itself.

Symbolic appearance imagery. This type of imagery was less common, but was used across the tests. When discussing the integral test, the students generally used the appearance of the sequence to write the function for the integral. This was done by simply swapping any "n" in the sequence formula for an "x" in the integral. Next, several students discussed the comparison tests in terms of how related the symbolic expressions of the two sequences were. For example, Student A began explaining the direct comparison test by using the example of two series with sequences $1/(n^2 + 2)$ and $1/n^2$. She stated that she used these two because, "You're just comparing something that is similar." Also, the students frequently invoked sequences of the form $[]^n$ as examples for the root test, and sometimes used sequences with factorials for the ratio test. These suggest the usage of symbolic appearance to match some series with certain tests.

Spatial location imagery. Students frequently used a hand or finger to locate a number, a sequence, or a series in the space in front of them. While this type of visualization occurred throughout the interviews, it occurred more often when students discussed the comparability of sequences or series. For example, Student B described the limit comparison test by stating, "If it's not zero or infinity, you know they are related to each other enough that they're going to do the same thing." As she said this, she brought her fingers together at two locations on the table that were close to each other. Later, she stated, "If it goes to zero, if it goes to infinity, it gives you the same result, but you can't compare them because they're too different." In this case, she cupped her hands in two different locations that were much farther apart.

Summaries of Visual Imagery Used

We next provide summary counts for how often each type of imagery was used by the students while discussing each convergence test (Table 2). The percentages are out of the total number of instances coded for *that* particular convergence test. Each test had certain types of imagery used more frequently than other types, and we shaded in gray any having at least 20%. We note that within each test, there was a reasonable amount of consistency across the students in terms of which types of visual reasoning they employed for that convergence test. Thus, we consider it sufficient for this report to only show the aggregate data across all students.

Table 2. Frequencies and percentages of each type of visual reasoning for each convergence test.

	Divergence (n = 103)	Integral (n = 97)	D. Comp (n = 69)	L. Comp (n = 79)	Ratio (n = 160)	Root (n = 86)
Concrete	15 (15%)	39 (40%)	12 (17%)	7 (9%)	44 (28%)	9 (10%)
Pattern	0 (0%)	0 (0%)	2 (3%)	4 (5%)	13 (8%)	7 (8%)
Formulae	9 (9%)	4 (4%)	5 (7%)	13 (16%)	14 (9%)	7 (8%)
Kinesthetic	46 (45%)	22 (23%)	12 (17%)	18 (23%)	52 (33%)	18 (21%)
Dynamic	13 (13%)	2 (2%)	6 (9%)	2 (3%)	11 (7%)	23 (27%)
Sym App	3 (3%)	14 (14%)	9 (13%)	10 (13%)	7 (4%)	18 (21%)
Spatial Loc	17 (17%)	16 (16%)	23 (33%)	25 (32%)	19 (12%)	4 (5%)

Using Visual Imagery to Reason about the Convergence Tests

In this final results subsection, we now describe how these types of visual imagery appeared to influence the students' reasoning about each of the convergence tests.

Divergence test. Kinesthetic imagery seemed to help the students reason about the divergence test. Students B, C, E, F, and G visualized active summations of the sequence terms and gestured grabbing or gathering the terms into a total. Many also raised their hand up and to the right to talk about how adding up infinitely non-negligible terms would result in a total that diverged. However, the dichotomous nature of the "upward" divergence and "downward" convergence motions may have caused occasional confusion. Student I used this dichotomous visualization to conclude that *any* sequence with limit zero would have a convergent series. She stated that the purpose of all other tests was simply "making it easier to take the divergence test."

For this test, students also used the concrete imagery of a list of sequence terms. However, a problematic component of this imagery arose for Students D and I. These students envisioned sequences as having a "final term" that is the value of the limit (cf. Davis & Vinner, 1986). For example, Student D was discussing a sequence with limit "e," and said, "The last number that we're going to add is e. So, there is a possibility that all of the numbers before that might be small enough so that the whole thing doesn't go to infinity, or the series doesn't go to infinity."

Integral test. A significant portion of the students' imagery used for the integral test involved the type of concrete image shown in Figure 1. However, importantly, while all of the students were able to produce this type of image, most of them did not know how to reason about it. In fact, only Student C was able to describe how the areas of these rectangles could be used to show that $\sum_{n=2}^{\infty} a_n \leq \int_1^{\infty} f(x)dx \leq \sum_{n=1}^{\infty} a_n$. Several students' reasoning simply focused on the visual aspect of how the heights of the rectangle "followed" the graph. For example, Student C stated, “It’s drawing a line through all your little points, and if that line converges to something, then that means that your series will converge.” Thus, it was the simple visual similarity between the curve and rectangle tops that became the salient feature, rather than the comparison between the area under the curve and the area of the rectangles. Student H even compared the rectangles to “a kind of histogram-like thing,” which likely elicited ideas about how statistical histograms can sometimes be seen as an approximation to a population distribution curve.

Direct comparison test. The students were generally successful in reasoning about this test, mostly through identifying “comparable” series and sequences. The students often spatially located two series near each other when describing compatibility, or far from each other when describing incompatibility. Symbolic appearance was a key reasoning tool in identifying comparable series. The students often described two series as comparable if the sequences had the same “leading term” symbol, such as $1/n$ and $1/(n–1)$, or $1/n^2$ and $1/(n^2+1)$. One issue with reasoning for this test was a failure to distinguish between comparing sequences, $a_n \leq b_n$, versus comparing series, $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$ (cf. Earls, 2017). For example, Student F stated, “So, when you compare to a series that you know diverges, if it’s greater than that series, then you know that this one also diverges.” Student F compared the series, rather than the sequences as stipulated by the test. Focusing on symbolic appearance may have led some students to overlook the differences between sequences and series, as well as which one the condition of this test uses.

Limit comparison test. For this test, as with direct comparison, the students also frequently reasoned about “comparability.” Symbolic appearance imagery again helped the students know what comparisons to make, with Student H even suggesting that the point was to find a “prettier” version of the given sequence. However, the students had more difficulty reasoning about why the test actually works. Some knew that a finite, non-zero result meant that the two series would do the same thing, but they were unsure how to justify it. Only Students B and C gave possible reasons. They loosely argued, based on kinesthetic imagery, that as the sequences “went” to infinity, their terms became more similar to each other. The spatial location imagery was also at times helpful, but at other times not. For example, Student F justified the result “infinity means inconclusive” by saying, “If you compare them and they go to infinity [places one hand off to the right and the other off to the left], then it kind of seems like they’re not similar at all.” Infinity seemed to have induced a spatial arrangement that led Student F to correctly believe the two sequences under consideration were not “similar.” However, he then struggled to use the same type of reasoning for the result “zero means inconclusive.” He said, “But zero, I guess, they’re also not similar [hesitantly placing hands apart]… I don’t know.” It appeared that the use of the word “zero” felt inconsistent with placing two objects far apart, which interfered with Student F’s reasoning and led him to doubt his conclusion.

Ratio test. All of the students struggled to reason about the ratio test. Most struggled to do more than describe the computational process of using the test. Four students (B, E, F, and G) made incorrect connections to the limit comparison test, based on the similar symbolic appearance of a_n/b_n and a_{n+1}/a_n. Problematically, this led Student E to claim that the ratio test results “should be the same as the limit comparison test.” Based on the analogy to limit comparison, Students E and G used dynamic imagery to incorrectly imagine the term a_{n+1} as representing some sort of graphical shift of the a_n sequence. Student E imagined a_{n+1} as an “upward” shift of a_n, and Student G described a_{n+1} as, “If you add one, you shift the graph, basically, to the left.” On the other hand, computationally, students tended to use factorial examples, suggesting that that symbolic appearance was at least beneficial in determining one type of series for which it is helpful to *use* this test. Lastly, two students (D and H) used a problematic concrete image of a graph with a horizontal line at $y = 1$ to reason about the ratio test results. They incorrectly imagined that if the sequence converged to anything below that line, the series converged, which seemed to conflate sequences and series convergence (cf. Earls, 2017).

Root test. The root test was also quite difficult for the students to reason about. Students B and C, however, made meaningful comparisons to the geometric series based on the symbolic similarity of the $[]^n$ format. This reasoning is used in one proof of the root test. Thus, in this case, symbolic appearance played a useful role in these students' reasoning. Other students attempted to use dynamic imagery to help provide some rationale for the test. The students saw the root test as getting rid of the exponent, *n*, allowing them to focus in on the "inside" of the sequence. They were not able to provide a rationale for why the test had the results that it did, though. Yet, Student D did use dynamic imagery in a different way, explaining that *n*-th roots of numbers less than one produce larger values. He described a sequence converging to zero and imagined a term for a very large *n*. "That would be a really small value. And then when we take the *n*-th root of that, that will increase the value [spreads hands apart]. If that increased value is between 0 and lesser than 1, that means that it went to 0 fast enough." Lastly, pattern imagery also played a useful role for most students by assisting in the quick recall of the appropriate test results.

Discussion

In this study, we saw differences in the type of imagery students relied on to reason about each test, such as kinesthetic imagery for the divergence test, concrete imagery for the integral and ratio tests, and spatial location imagery for the two comparison tests. Further, we saw that some tests were easier to reason about, like the divergence and direct comparison tests, and others were more difficult to reason about, like the integral, ratio, and root tests. We identified particular aspects of visualization that helped or hindered the students' reasoning. For example, symbolic appearance helped students reason about the direct comparison and root tests, but then seemed problematic for reasoning about the ratio test. Dynamic imagery helped students focus their attention on the relevant parts of a symbolic expression. Concrete imagery was useful in imagining "sizes" or sequence terms, but was problematic when that imagery was not well understood, like incorrectly "completing" the image of a sequence by including a "last term." The common picture used for the integral test also turned out to be poorly understood by the students. This result has implications for how instructors may wish to use concrete imagery for convergence tests, such as using formative assessment to ensure that students understand what those images are meant to be representations of. Lastly, spatial location imagery may have been helpful for students in cognitively managing the various objects being referred to in a test. This lesser-studied type of visualization may need more attention in future work.

Our study has many connections to research on sequences, series, and convergence. The sequence list imagery is closely related to the SEQLIST conception described by McDonald et al. (2000). Martinez-Planell and Gonzales (2012) argued that a SEQLIST conception is less productive for understanding series convergence than if the sequence is understood to be a function from the natural numbers to the reals. Our results in some ways agrees with this claim, but in other ways disagrees, since this imagery was at times helpful for the students. However, it is true that the sequence image with a "last term" is certainly problematic (see Davis & Vinner, 1986). For this misconception, our results may actually indicate a possible *imagery*-based origin.

The concrete imagery of a continuous graph for the integral test also shows that students seemed to have internalized this common image (see González-Martín et al., 2011). However, the students' understanding of this image seemed far from the intention of the picture. Finally, we can see that series of the form $\sum_{n=1}^{\infty} 1/n^m$ for m = 1, 2, or 3 were commonly used and might indeed be "prototypical" examples of series, as mentioned by Alcock and Simpson (2002).

References

Alcock, L., & Simpson, A. (2002). Definitions: Dealing with categories mathematically. *For the Learning of Mathematics, 22*(2), 28-34.

Alcock, L., & Simpson, A. (2004). Convergence of sequences and series: Interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics, 57*(1), 1-32.

Davis, R., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *The Journal of Mathematical Behavior, 5*(3), 281-303.

Earls, D. (2017). Second semester calculus students and the contrapositive of the nth term test *Proceedings of the 20th annual special interest group of the Mathematical Association of America on research in undergraduate mathematics education*. San Diego: SIGMAA on RUME.

Earls, D., & Demeke, E. (2016). Does it converge? A look at second semester calculus students' struggles determining convergence of series. In T. Fukawa-Connelly, N. E. Infante, M. Wawro, & S. Brown (Eds.), *Proceedings of the 19th annual Conference on Research in Undergraduate Mathematics Education* (pp. 111-124). Pittsburgh, PA: SIGMAA on RUME.

González-Martín, A., Nardi, E., & Biza, I. (2011). Conceptually driven and visually rich tasks in texts and teaching practice: The case of infinite series. *International Journal of Mathematics Education in Science and Technology, 42*(5), 565-589.

Mamona-Downs, J. (2001). Letting the intuitive bear on the formal: A didactical approach for the understanding of the limit of a sequence. *Educational Studies in Mathematics, 48*(2-3), 259-288.

Martinez-Planell, R., & Gonzales, A. C. (2012). Students' conception of infinite series. *Educational Studies in Mathematics, 81*(2), 235-249.

McDonald, M. A., Mathews, D., & Strobel, K. (2000). Understanding sequences: A tale of two objects. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education* (Vol. IV, pp. 77-102). Providence, RI: American Mathematical Society.

Presmeg, N. C. (1986). Visualisation in high school mathematics. *For the Learning of Mathematics, 6*(3), 42-46.

Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present, and future* (pp. 205-235). Rotterdam, The Netherlands: Sense Publishers.

Tall, D. O. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity, and proof. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 495-511). New York, NY: Macmillan.

Tall, D. O., & Vinner, S. (1981). Concept image and concept definition in mathematics, with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.