Modeling the Spread of Ideas in an Inquiry-Oriented Classroom

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In this study, we model the spread of student understanding of linear combinations in an Inquiry-Oriented Linear Algebra (IOLA) class based on video analysis. Methods adapted from modeling biological systems were used to estimate the rate of spread of Process-level understanding of linear combinations, measured according to Action-Process-Object-Schema (APOS) theory. The amount of time required for all students to achieve Process-level understanding was also estimated.

Keywords: Inquiry-Oriented, Mathematical Modeling, APOS, Linear Algebra

Introduction

Over the past thirty years, there has been a movement to use active learning in mathematics instruction or "instructional activities involving students in doing [mathematics] and thinking about what they are doing" (Bonwell & Eison, 1991, p. iii). One instructional design theory is Realistic Mathematics Education (Freudenthal, 1991), which focuses on having students discover mathematical concepts through guided reinvention (Gravemeijer, 2004). Here we examine the spread of ideas in an Inquiry-Oriented Linear Algebra (IOLA) classroom. The tasks used in this study were developed from a larger instructional design project (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012). We chose to focus on the spread of the idea of linear combination during the first two class periods of the course, roughly 120 minutes of instructional time.

We used Action-Project-Object-Schema (APOS) theory as a framework to determine whether or not a student "understood" the idea of linear combination (Arnon, et al., 2014). At an Action conception, students are concerned with an external transformation of objects. At the Process level, this activity is interiorized so the student can run through it mentally. At the Object level, that Process is encapsulated into a static entity. At the Schema level, a student is able to coordinate Processes and Objects and thereby act on them. We used Arnon et al.'s (2014) genetic decomposition for spanning set, which included a partial decomposition of linear combination, to determine if a student "understood" linear combinations. In particular, students demonstrating at least a Process-level conception of linear combination were classified as "understanding", specifically:

> Interiorization of [vector addition and scalar multiplication] yields a Process for constructing a new vector which is an element of the vector space, that is, the Process of constructing a linear combination. The reversal of this Process allows the student to verify if a given vector can be written as a linear combination of a given set of vectors. Students who show they have constructed these processes are considered to have a Process conception of a linear combination (p. 36-37).

We operationalized "understanding" linear combinations as verbally articulating a solution method indicating a Process-level conception of linear combination. This includes the ability to add two vectors multiplied by scalar weights and being able to interpret the procedure in at least two contexts (e.g., system of equations, vector equations, graphically).

To follow the spread of the idea of linear combination through an IOLA classroom, we generated two research questions modeled on the language of mathematical biology: (1) what is the infectivity rate for students discussing linear combinations in an IOLA classroom and (2) how long should one expect it to take for all students in the course to reach a Process-level conception of linear combinations?

Methods

Data Collection

We watched videos of the first two days of class that recorded three tables of students and captured whole class discussion. We defined a contact as a verbal communication of mathematically relevant content related to linear combinations. We did not consider written work on paper without any verbal explanation to be a contact. Each of the tables' discussions were analyzed for contact rates between members of the table as well as for contacts coming from outside the table. Outside contacts were considered to come from the teacher or the "infected" (understanding) members of the classroom who addressed the whole class. We recorded both contacts from "infected" to "non-infected" (not yet understanding) persons and when we had evidence that a student understood according to our definition.

The coding of the videos for contacts and indications of student understanding was done iteratively. Two researchers independently watched each video and then compared conclusions to check for consistency. After the contacts were counted, we calculated the mean for each student across each researcher's numbers. We only considered contacts for the fifteen individuals at the three tables that we closely observed and assumed their interactions were representative of those in the 35 person class. Due to the variation in duration and types of interactions (e.g. teacher-to-table, student-to-whole class, group member-to-group member), contacts were weighted differently. Contacts between group members at tables were given weight 1 and contacts from the teacher or students from other groups in the class were given weight 0.5 because the group setting allowed for more opportunities for students to engage in each other's reasoning, leading to higher quality interactions. The weighted contact values and times when students understood linear combinations are found in Table 1. Students' names have been replaced with pseudonyms.

Model Development

We developed an Ordinary Differential Equation (ODE) model and a Continuous Time Markov Chain (CTMC) model for following the spread of understanding. The CTMC was chosen because it better models systems with lower population size than ODE models.

ODE model. We used the Susceptible-Infected (S-I) ODE model because we considered all students entering the classroom as being "susceptible" to understanding the concept of linear combination, and we assumed that students who understood the concept did not forget it. Thus there is no "recovery" from understanding, and individuals do not become susceptible again. We assumed that the teacher was the initial infected person, such that $I(0) = 1$ and $S(0) = N - 1$. This yields the following system of equations:

$$\frac{dS}{dt} = -\frac{\beta SI}{N}, \qquad \frac{dI}{dt} = \frac{\beta SI}{N}, \qquad N = S + I$$

The susceptible individuals are the students in the classroom that do not understand yet, and the infected individuals are the individuals that do. The total population, $N = 35$, is the number of individuals participating in the classroom (including the teacher). We used the average number of people from both days as the total population and assumed the size of the population was constant. The parameter $\beta > 0$ is the transmission rate, calculated from the product of the average number of contacts between susceptible and

Table 1. Students' contacts with understanding persons by researcher and the number of minutes until students displayed Process level conceptions of linear combinations.

Table	Student	Weighted R1	Weighted R2	Ave Weighted	Minutes to "Infection"
1	Philip	10	9.5	9.75	
1	Sue	10	9.5	9.75	
1	Houston	10	9.5	9.75	
1	Shane	2.5	2	2.25	
1	Alice	2.5	2	2.25	
2	Gavin	1.5	0.5	1	12
2	Karl	4.5	8.5	6.5	23
2	Carly	13	21	17	69
2	Mark	13	32	22.5	76
2	Colin	37	67.5	52.25	
3	Devin	2	2	2	19
3	Ahsan	4.5	12	8.25	70
3	Ernesto	27	23	25	
3	Kurt	27	23	25	
3	George	27	23	25	

infected individuals per time and the probability of infection per contact. Larger β values indicate a higher probability of "catching" understanding than smaller β values. We assumed that everyone in the classroom had the capacity to understand linear combinations but students did not enter with that understanding, so everyone except the teacher started in the susceptible class.

CTMC Model. We also considered a stochastic model using CTMC (Allen, 2008). We define our CTMC on $t \in [0, \infty)$ where $t = 0$ is the time when the class started working with mathematical content on the first day. The states $S(t)$ and $I(t)$ are discrete random variables; that is $S(t), I(t) \in \{0, 1, 2, ..., N\}$ where $N = 35$. Here we are largely concerned with the dynamics for $t \in [0, 120]$, the classroom time in which the students engaged in the tasks. Each random variable depends on the probability functions $p_i(t) = \text{Prob}\{I(t) = i\}$. We assume the Markov property holds: for any sequence of real numbers t_i for $i = 0, 1, ..., n+1$, where $0 \leq t_0 < ... < t_{n+1}$, $\text{Prob}\{I(t_{n+1}) : I(t_0), I(t_1), ..., I(t_n)\} = \text{Prob}\{I(t_{n+1}) : I(t_n)\}$. Thus the transition probability at time t_{n+1} only depends on the most recent time step, t_n.

We consider the transition probabilities to be defined for time intervals of length $\Delta t = 1$ minute. We assume this is sufficiently small so at most one person is infected in a single time step. Under this assumption, the transition probabilities are as follows:

$$p_{ji}(\Delta t) = \begin{cases} \frac{\beta i(N-i)}{N}\Delta t + o(\Delta t) & j = 1 \\ 1 - \frac{\beta i(N-i)}{N}\Delta t + o(\Delta t) & j = 0 \\ o(\Delta t) & j \neq 0, 1 \end{cases}$$

Case $j = 1$ gives the probability that one individual transitioned from not understanding to understanding in a time step, $j = 0$ is the probability of no change, and $o(\Delta t)$ is the probability that more than one individual is infected at one time (assumed to be 0). To simplify the probability expression, let $\beta i(N - i) = b(i)$. Then $b(i)$ represents the infection rate of individuals at time Δt. The simplified version of the model is as follows:

$$p_{ji}(\Delta t) = \begin{cases} b(i)\Delta t + o(\Delta t) & j = 1 \\ 1 - b(i)\Delta t + o(\Delta t) & j = 0 \\ o(\Delta t) & j \neq 0, 1 \end{cases}$$

Simulation Model

We used the values in Table 1 to determine the average number of contacts for each of the students at the three tables. We then divided this by 118, the total number of minutes of instruction across the two days. This produced $\beta = 0.1233$. We also found the "best fit" β value according to the deterministic ODE model, which was $\beta = 0.0426$ according to nlinfit in Matlab. The code used to generate figures was adapted from a Mathematical Modeling course (L. Childs, personal communication, March 23, 2017) and Allen (2008).

Results

Analytical Results

There are two steady states for the ODE system: "disease-free" $(N, 0)$ and "endemic" $(0, N)$. We want understanding to be endemic; that is, we want the entire population to understand. To study the steady states, we used the Jacobian method which yielded eigenvalues of $J(0, N)$ of $\lambda_1 = 0$ and $\lambda_2 = -\beta$. We note that $\beta > 0$; thus $(0, N)$ is always stable, provided there is one infected person initially. Using the same method it is easy to see $(N, 0)$ is always unstable when there is an initial infected person.

For the stochastic model, we consider the inter-event time, T_i, which is the expected time until everyone is infected. $T_i = -\frac{\ln(1-U)}{b(i)}$ is an exponentially distributed random variable with parameter $b(i)$ (defined above) and uniform random variable U on $[0, 1]$. We use this to calculate the expected time until everyone understands, which is equivalent to reaching the endemic steady state. Recall that $I(0) = 1$. Thus the first state is one infected person. We know the expected time to reach the next state (two infected people) is T_1. Let W be the time it takes to get from state 1 to state 35. Then $W = \sum_{i=1}^{34} T_i$. Because T_i is exponentially distributed with parameter $b(i)$, the expected value is given by $\mathbb{E}(T_i) = \frac{1}{b(i)}$. Furthermore, because $\mathbb{E}$ is a linear operator, $\mathbb{E}(W) = \sum_{i=1}^{34} \frac{1}{b(i)}$.

Simulation Results

In Figure 1, we see a comparison of the scaled data points extrapolated from the observed fifteen students to the whole class, the deterministic solution to the ODE, and 20 stochastic paths. In Figure 1A the first three data points are close to the deterministic solution and in the midst of the stochastic paths. However the three later data points are far below the deterministic and stochastic paths. Although $\beta = 0.1233$ is based in our classroom observations, it does not appear to model when students obtain Process conceptions of linear combination, though it did perform better than the non-weighted version. The non-weighted version, based on counting contacts from the teacher, students from other tables, and students from the same table equally (Figure 1B) used $\beta = 0.1870$, which fit even worse.

Unlike the curves in Figure 1, which matched the early data points well but missed later points badly, the curve in Figure 2, which is based on the "best fit" $\beta = 0.0426$, misses most of the early data points but fits the later points well. Additionally, at 120 minutes, not all students are "infected". This fits the data better because a number of our students did not appear to obtain a Process conception of linear combination by the end of the 118 minutes of class time we observed.

Furthermore, using $\beta = 0.1233$ produced an expected wait time for all students to obtain a Process conception of linear combination of 67 minutes, roughly half the time we observed the class. When using $\beta = 0.0426$, which represents a reduced probability of infection, the expected time for all students to understand was 193 minutes. This amount of time seems more plausible because only six of the fifteen students we observed in detail appeared to have obtained Process level conceptions by the end of the second class period.

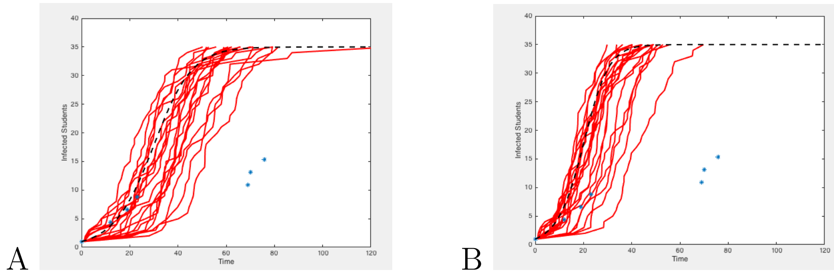

Figure 1. These graphs use $N = 35$, $I(0) = 1$, and a time step of 1 minute. The black dashed lines represent the deterministic solution to the ODE. The 20 red lines on each graph are stochastic simulations. The blue stars represent the data points obtained by scaling the fifteen students' data to the full class of 35 people. The left simulations (A) ran with $\beta = 0.1233$ and the right simulations (B) used $\beta = 0.1870$.

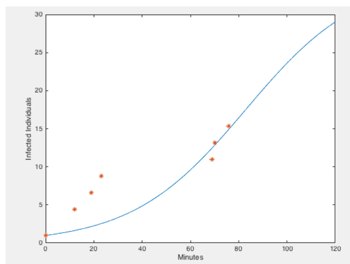

Figure 2. This graph uses $N = 35$, $I(0) = 1$, initial guess $\beta = 0.1$, and a time step of 1 minute. The blue line represents the number of infected individuals at a given time. The orange stars represent the data points obtained by scaling the fifteen students' data to the full class. The resulting curve results when $\beta = 0.0426$.

Discussion

Our goals were to determine the infectivity rate for students discussing linear combinations in an IOLA classroom and to determine the expected time for all students in the course to reach a Process-level conception of linear combinations. We determined that the infectivity rate $\beta = 0.1233$, based on creating weighted contact values, estimated more effectively than the non-weighted contact values, but still did not estimate very well. However, when we used maximum least squares to estimate the infectivity rate from the infection data, we obtained $\beta = 0.0426$, which produced more reasonable long term results, including requiring slightly more than another full period for full class understanding.

We acknowledge that this study and model have a number of limitations, including incomplete classroom data, discontinuous data collection, and a non-constant population of students. Specifically, while we have direct information about 15 students and the teacher, we have limited information about the other 19 students. Even in the data on the 15 students, we were, if anything, a bit conservative on saying a student understood the concept of linear combination. It is possible that students who did not speak as often in the groups also understood; we simply did not have enough evidence to be sure that they did.

Future work could include further refining β through more extensive data collection. This could involve observing subsequent class periods to see if 193 minutes was sufficient time for everyone to understand linear combinations. Alternatively, future data collection could involve setting up cameras at more students' desks. We could also refine the weighting of contacts. Perhaps group contacts should be weighted even more heavily or perhaps specific "infected" individuals, like the teacher, have a greater impact than other individuals. Finally, we could consider implementing an age structure or risk structure model instead of basing the model on Markov Chains.

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