A Study of Calculus Students' Solution Strategies when Solving Related Rates of Change Problems

Thembinkosi P. Mkhatshwa
Miami University

Steven R. Jones
Brigham Young University

Contributing to the growing body of research on students' understanding of related rates of change problems, this study reports on the analysis of solution strategies used by five calculus students when solving three related rates of change problems where the underlying independent variable in each problem was time. Contrary to findings of previous research on students' understanding of related rates of change problems, all the students in this study were able to translate prose to algebraic symbols. All the students had a common benchmark to guide their overall work in one of the tasks but no benchmark to guide their overall work in the other two tasks. Three students exhibited weaker calculational knowledge of the product rule of differentiation. Directions for future research and implications for instruction are included.

Key words: related rates, implicit differentiation, problem solving, calculus education

Related rates of change problems form an integral part of any first-year calculus course. However, there have been relatively few studies that have examined students' reasoning about related rates of change problems. Engelke (2007) argued that there is a dearth of research that examines how students understand and solve related rates of change problems in introductory calculus. Findings of a comprehensive review of literature on students' understanding of various topics in college calculus by Speer and Kung (2016) indicate that studies on related rates of change are scarce. Of the few studies involving related rates problems, Piccolo and Code (2013) found that students had computational difficulties when calculating derivatives involving more than one time-dependent variables. Engelke (2007) described several beneficial components for successful solutions, including drawing diagrams, determining functional relationships (algebraic equations), and checking the answer. She also added that proficiency with the chain rule helped the students make sense of the problem context and the components of the functional relationship. Other studies have focused more on how students read and interpret the problem statement (Martin, 2000; White & Mitchelmore, 1996). White and Mitchelmore found that students struggled to "symbolize," or mathematize, the related rates problems, often being unsure as to how to use all the given information in a related rate problem. Martin also found that students struggled to convert the written prompt into a mathematical structure on which the students could operate.

While these studies have provided beneficial information about how students set up and solve related rates of change problems, there is still much to be explored about the specific difficulties that limit students' success when solving such problems, which is the motivation for this study. Thus, to build on these studies, we intend to explicitly examine students' understanding of the key steps that are generally involved in the process of solving related rates of change problems where the underlying independent variable is time. In particular, our study was guided by the following research question: What do calculus students' solution strategies when solving related rates of change problems reveal about the difficulties that limit students' success when solving such problems?

Related Literature

For a working definition, a mathematical task is said to be a related rates of change problem (abbreviated as "related rates problem") if it involves at least two rates of change that can be related by an equation, function, or formula. As noted earlier, research on students' understanding of related rates of change is sparse. Piccolo and Code (2013) analyzed Calculus I students' written responses to related rates problems at a large research university. This analysis revealed that the students were successful in performing the early steps of solving the problems (e.g., identifying the quantities involved in the problem and finding an equation that relates these quantities). However, using implicit differentiation was a major issue for the students' success in the problems. More specifically, Piccolo and Code reported that implicitly differentiating functions with several time-dependent variables, with respect to time, was problematic for a majority of the students. Consequently, most of the students were unsuccessful in solving the problems they were given. Piccolo and Code asserted that students' difficulties with solving related rates problems stems from a lack of facility with the process of differentiation, rather than a misunderstanding of the physical context of such problems.

Engelke (2007) used a teaching experiment, consisting of six teaching episodes, to examine how calculus students understand and solve related rates problems. Engelke argued that knowledge of the chain rule appeared to help the students in solving the problems they were given. This researcher reported that the students had difficulty imagining each variable in the problems as a function of time especially when time was not explicitly mentioned in the problem statement. Engelke further proposed a framework for analyzing students' work when solving related rates problems. Details of this framework are provided in the next section.

Martin (2000) analyzed students' responses in a problem-solving test containing several items assessing the students' ability to solve related rates problems in an introductory calculus class. Martin reported that overall performance was poor, and claimed that "the poorest performance was on steps linked to conceptual understanding, specifically steps involving the translation of prose to geometric and symbolic representations" (p. 74). White and Mitchelmore (1996), who studied the conceptual knowledge of 40 undergraduate mathematics majors when solving four application problems (including two related rates problems) at the level of introductory calculus, reported similar results. White and Mitchelmore also found that the students tended to replace unfamiliar variables with either x or y, an idea that has come to be known as the "x, y syndrome" (p. 89).

Conceptual Framework for Related Rates

This study uses Engelke's (2007) framework, which characterizes the phases involved in solving related rates problems in calculus. The framework emerged from interviews, using a think-aloud protocol, with three mathematics professors who were solving three related rates problems similar to the tasks we used in this study. The framework lists five phases that one follows when solving related rates problems. These phases are: (1) draw and label a diagram, (2) determine a meaningful functional relationship, (3) relate the rates of change, (4) solve for the unknown rate of change, and (5) check the answer for reasonability. However, in our study, we realized the need to add an additional "orienting" phase, because the students often spent time simply acquainting themselves with the problem context. We label this the "zero-th" phase, (0) orienting to the problem. The following is a description of each of these phases.

The orienting phase consists of the solver carefully reading the problem (aloud or silently), with the goal of identifying what the solver considers to be important information. More specifically, the solver identifies given quantities and the required quantity (the unknown rate).

Then, in the diagram phase, the solver draws and labels a diagram illustrating the relationship of the quantities in the problem. In Task 1 (methods section), for example, the solver may draw a right triangle where the horizontal leg of the triangle represents the distance between the westbound plane and the airport, the vertical leg of the triangle represents the distance between the northbound plane and the airport, and the hypotenuse of the triangle represents the distance between the two planes.

In the functional relationship phase, the solver constructs a meaningful relationship (algebraic equation) between the quantities he/she identified while orienting himself/herself with the problem or while drawing a diagram. In the case of Task 1, the solver may use the Pythagorean Theorem, $x^2 + y^2 = z^2$, where x is the distance of the westbound plane from the airport at any point in time, y is the distance of the northbound plane from the airport at any point in time, and z is the distance between the two planes at any point in time. During the relate the rates phase, the solver implicitly differentiates with respect to a time variable the algebraic equation he/she identified as relating the quantities in the problem. The process of implicit differentiation results in the creation of a new equation that shows a relationship of the rates of change involved in the problem. In Task 1, the process of implicit differentiation with respect to a time variable t, would result in the equation, $2xdx/dt + 2ydy/dt = 2zdz/dt$, where dx/dt is the speed of the westbound plane at any point in time, dy/dt is the speed of the northbound plane at any point in time, and dz/dt is the rate at which the distance between the two planes reduces at any point in time.

In the solve for the unknown rate phase, the solver substitutes all the given quantities in the new equation and then solves for the required rate of change. In Task 1, this means solving for the value of z at a particular point in time when the quantities of x and y are known, using this value of z together with the known values of the quantities x, y, dx/dt, and dy/dt to solve for the unknown quantity, dz/dt. Finally, during the check the answer phase, the solver uses certain goals or benchmarks to guide their overall work. The goals or benchmarks include: (a) having a sense of knowing if the answer the solver found (the unknown rate) is higher or lower than would be expected, (b) expecting the answer to have a particular sign (positive or negative), and (c) expecting the answer to have particular units (e.g., miles per hour instead of miles). Checking the answer for reasonability in Task 1 may, for instance, mean having an awareness that the units of dz/dt should be in miles per hour since it is a rate and that the sign of the value of dz/dt should be negative as the distance between the two planes decreases over time.

Methods

This qualitative study used task-based interviews (Goldin, 2000) with five students. The interviews lasted about 45 minutes, on average, and contained three tasks:

> **Task 1:** Two small planes approach an airport, one flying due west at a speed of 100 miles per hour and the other flying due north at a speed of 120 miles per hour. Assuming they fly at the same constant elevation, how fast is the distance between the planes changing when the westbound plane is 180 miles from the airport and the northbound plane is 200 miles from the airport?
>
> **Task 2:** A leak from the sink is creating a puddle that can be approximated by a circle, which is increasing at a rate of 12 cm^2 per second. How fast is the radius growing at the instant when the radius of the puddle equals 8 cm?

Task 3: For the next problem, let me give you a little background on a formula that we will use. Suppose a gas is inside of a container. Many gases under normal conditions follow the "ideal gas law," $PV = kT$, where P is the pressure the gas exerts on the container, V is the volume of the container, T is the temperature of the gas, and k is a constant. P is measured in "atmospheres," V is measured in cubic meters, and T is measured in Kelvins. Kelvins is a lot like Celsius, except that it is scaled so that 0 means absolute zero (lowest possible temperature), which makes water's freezing point to be $273\ °K$. Do you have any question(s) about this formula, or any of the quantities [like temperature in Kelvins] before we proceed?

In a laboratory, an experiment is being done on a gas inside a large, flexible rubber balloon. For the experiment, the temperature of the gas is being heated at a rate of 8 degrees per second. At one point, when the temperature of the gas is $300\ °K$, the pressure is 1.5 atmospheres, the volume of the gas is one cubic meter, and the volume of the gas is growing at a rate of 0.01 m^3 per second. At that moment, is the pressure in the balloon increasing or decreasing? What is the rate of that increase/decrease?

The students worked through these tasks while the interviewer asked clarifying questions about their work. After the student concluded their work for each problem, the interviewer asked the following questions about the task and the content of their solutions: (a) Have you seen a problem like this before? (b) What did you need to do to solve this task? (c) What does your answer tell you? (d) What does the sign of your answer tell you? (e) What are the units for the rate you found? (f) What does each quantity throughout your solution [including amount quantities and rate of change quantities] mean? (g) What does each computational step mean in terms of the quantities? (h) Will your answer for this problem be the same for all points in time for this context? Most of the students' time was spent on Task 1 while the least amount of time was spent on Task 3. We remark that Task 3 was not a routine task to the students in that the students' prior experiences with related rates problems in course lectures and in the course textbook was limited to problems similar to Task 1 and Task 2.

Setting, Participants, and Data Collection

The study participants were five undergraduate students (pseudonyms Ben, Bill, Jake, Nick, and Tim) at a research university who were enrolled in a traditional calculus I course in the summer of 2017. The course met twice a week (each meeting lasted for 2 hours and 45 minutes) for a duration of 12 weeks. The students were recruited via an official class roster obtained from their professor. The students were chosen based on their willingness to participate in the study. The students were familiar with the key ideas examined in the three tasks (instantaneous rate of change and the process of implicit differentiation) from course lectures and the course textbook. Three of the participants (Ben, Nick, and Tim) were Business/Economics majors while the other two students (Bill and Jake) were Engineering majors. Two students (Ben and Tim) had taken a high school calculus course prior to participating in this study. At the time of the study, three of the students were sophomores, one student was a junior, and the other student was a senior. The cumulative grade point averages (GPAs) of the five students had a mean of 2.02 on a 4.0 scale and a standard deviation of 0.71, suggesting that these were low performing students. As we would discuss in the concluding section of the paper, analysis of interviews conducted with this sample of students were both surprising and interesting at the same time. Data for the study consisted of transcriptions of audio-recordings of the task-based interviews and work written by the five students during each task-based interview session. When transcribing the audio recordings of the interviews, we used video recordings of the interviews to check what students were referring to when they pointed at something during the interviews in cases when such information could not be easily obtained from work written by the students during the interviews.

Data Analysis

Data analysis was done in two stages. In the first stage, we used a priori codes, consisting of the five phases from Engelke's (2007) framework, plus the additional "orienting" phase we included. More specifically, we carefully read through each interview transcript and coded instances where each student reasoned about: (0) how they interpreted the problem, (1) drawing diagrams illustrating what is happening in each task, (2) constructing algebraic equations relating the quantities in each task, (3) differentiating the equations, (4) solving for the unknown rate, and (5) checking the answer (unknown rate) for reasonability. In the second stage of analysis, we looked for patterns in each of the codes identified in the first stage of the analysis. These patterns included trends in the students' understandings, or difficulties they had in connection with each of the phases of the related rates framework. The common understandings or difficulties in students' reasoning found in the second stage of our analysis provided answers to our research question.

Results

Analysis of the data revealed that: (1) all the students were able to translate prose to algebraic symbols, (2) all the students had a common benchmark to guide their overall work in the first task, (3) students typically had no benchmark to guide their overall work in Tasks 2 and 3, and (4) three students exhibited difficulties with the product rule of differentiation.

Translating Prose to Algebraic Symbols

All of the students successfully identified the appropriate algebraic equation that shows how the quantities in each of Task 1 and Task 2 are related (Task 3 already had a formula given to the students). That is, translating the text of each task into algebraic symbols, and relating these symbols using the appropriate equation, was not problematic for the students. As previously noted in the methods section, we argue that this may have been because the students had been shown how to solve problems similar to Task 1 and Task 2 through examples that were given during course lectures. Four of the students provided reasonable rationales for using the Pythagorean Theorem in Task 1. Bill is representative of this group of four students. When asked how he moved from the "right triangle," which he claimed was a "picture" of what is happening in Task 1, to the equation $x^2 + y^2 = z^2$, Bill stated that the two planes "are heading due west and due north, it's a right triangle, so use the Pythagorean Theorem [pointing at the equation $x^2 + y^2 = z^2$]." When probed on what the Pythagorean Theorem meant in terms of x, y, and z, Bill indicated that it "relates all of them together." We remark that Bill correctly interpreted x, y, and z as distances that are measured in miles. More specifically, Bill interpreted x as the distance of the westbound plane from the airport, y as the distance of the northbound plane from the airport, and z as the distance between the two planes. When asked to elaborate on what the Pythagorean Theorem meant, Bill stated, "this side [pointing at one side of the triangle which he labelled x] squared plus this side [pointing at another side of the triangle which he labeled y] squared, equals this side [pointing at the hypotenuse of the triangle which he labeled z] squared." We argue that Bill recognized that by virtue of the two planes flying west and north respectively, the distance between the two planes is given by the hypotenuse of a right triangle. This is rightfully so because the two planes are assumed to be constantly flying at the same elevation in the description of the task.

One student, Ben, knew that he had to use the Pythagorean Theorem to relate the quantities in Task 1, because it was used in a similar problem that was solved in class. Ben, however, did not provide a convincing rationale for using the Pythagorean Theorem in this task. When asked on why he used the Pythagorean theorem in this task, Ben said "I don't know, I just wrote it up here [pointing at the equation, $a^2 + b^2 = c^2$] because I figured that the problem might have something to do with it." Ben added, "I don't know, it's a habit I have had since I started learning calc, or geometry for a long time, and so I actually implemented it to the solution right here [pointing at the equation, $x^2 + y^2 = z^2$ which was part of his solution]." All the students provided reasonable rationales for using the formula for the area of a circle as an equation that relates the quantities in Task 2. Common among these rationales was that the puddle is circular, as described in the task description.

Benchmarks used by Students to Guide their Overall Work

None of the students used any goals or benchmarks to guide their overall work in Task 2 and Task 3. More specifically, they did not mention any specific way of having a sense about whether or not their answers to these tasks were correct. However, in Task 1, all the students had an expectation that the required unknown rate had to be negative regardless of whether or not their work leading to the answer is correct. The following excerpt, illustrates how Nick, for example, determined the benchmark for his solution to Task 1.

Researcher: I noticed that c' came out as positive 160mph [Nick's Answer], what does that mean?

Nick: That it [c'] is increasing over time, like going up by increments by units so like if it's a hundred and sixty miles per hour now, over time it will go up to a hundred and seventy, a hundred and eighty, and it will get faster and faster. If it [c'] were negative, it would decrease. So in this case, I would assume it will be negative [changing 160mph to -160mph] because they [the two planes] are coming closer and closer [to each other], and they are getting near to the airport, so it [c'] will be decreasing.

We argue that Nick's claim that c' should be negative may have been a result of two things. First, Nick may have correctly determined that c' should be negative by reasoning about the context of the task, that is, the distance between the two planes is getting smaller as the planes approach the airport hence c' should be negative. Second, Nick could simply be recalling a justification for c' to be negative that was given by his calculus professor when he did a problem similar to Task 1 during course lectures.

Difficulties with the Product Rule

Three students (Tim, Bill, and Ben) incorrectly applied the product rule when differentiating the equation, $PV = kT$, in Task 3. Figure 1 shows how Tim used the product rule to differentiate $PV = kT$.

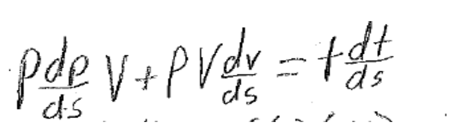

Figure 1. Tim's derivative of PV=kT.

By incorrectly applying the product rule, Tim concluded that the derivative of $PV = kT$ with respect to a time variable s (which he stated would be measured in seconds) would be

$PVdP/ds + PVdV/ds = TdT/ds$ instead of $PdV/ds + VdP/ds = kdT/ds$. Tim did not use the stated conditions in the task to determine the value of the constant k. Instead, he either discarded the constant k or substituted a “1” for it when taking the derivative of $PV = kT$. Bill and Ben treated k in a similar way when finding the derivative of $PV = kT$. Bill’s work (Figure 2) is representative of how these two students differentiated the equation, $PV = kT$.

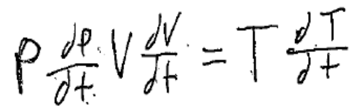

Figure 2. Bill's derivative of PV=kT.

Bill differentiated the equation, $PV = kT$, implicitly with respect to the time variable t which he said would be measured in “seconds”. As can be seen in the solution in Figure 2, Bill recognized that he had to use the product rule to find the derivative of $PV = kT$. He, however, incorrectly applied the product rule. More specifically, Bill’s derivative of PV is $PdP/dt * VdV/dt$ instead of $PdV/dt + VdP/dt$. His derivative of kT is TdT/dt instead of kdT/dt. The other two students (Nick and Jake) did not make any attempt of finding the derivative of $PV = kT$ or let alone mention that they have to find one. Instead, they systematically guessed the answer to the question of whether or not the pressure of the gas inside the balloon is increasing or decreasing.

Discussion and Conclusions

Contrary to findings of previous research (Martin, 2000; White & Mitchelmore, 1996) on students’ understanding of related rates problems, findings of this study indicate that translating prose to algebraic symbols was not the problematic part of the process for the students in this study. More specifically, the students in our study did not have any difficulty with symbolizing, algebraically, the quantities in each task in addition to finding an appropriate equation that relates the quantities mentioned in each task. This result is more striking given that the students in our sample were weaker overall, meaning that even these weaker students did not have issues with these parts of the process.

Interestingly, the students in this study expressed a common benchmark to guide their overall work in one of the tasks but did not express any benchmark to guide their overall work in the other two tasks. In particular, all the students indicated that the unknown rate they were trying to find in Task 1 (how fast the distance between the two planes is changing) had to be negative since the distance between the two planes decreased as the planes approach the airport. Future research might examine the role of problem context in students’ use of benchmarks to guide their overall work when solving related rates problems. Finally, students’ reasoning about the non-routine task (Task 3) revealed that most of the students in this study had difficulty using the product rule. As a recommendation, calculus instructors may need to help students develop greater facility with procedures such as the chain rule, product rule, and quotient rule prior to applying these rules when solving contextualized related rates problems (cf. Engelke, 2007). As a limitation of our study, we note that because the participants are five students from the same class, the results may not generalize.

References

Engelke, N. (2007). *Students’ understanding of related rates problems in calculus*. (Doctoral dissertation). Arizona State University, Arizona.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517-545). Mahwah, NJ: Laurence Erlbaum Associates.

Martin, T. (2000). Calculus students' ability to solve geometric related-rates problems. *Mathematics Education Research Journal*, *12*(2), 74-91.

Piccolo, C., & Code, W. J. (2013). Assessment of students' understanding of related rates problems. In S. Brown, G. Karakok, K. H. Roh, & M. Oehrtman (Eds.), *Proceedings of the 16th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp. 607-609). Denver, Colorado.

Speer, N., & Kung, D. (2016). The complement of RUME: What's missing from our research? In T. Fukawa-Connelly, N. Infante, M. Wawro, & S. Brown (Eds.), *Proceedings of the 19th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp. 1288-1295). Pittsburgh, Pennsylvania.

White, P., & Mitchelmore, M. (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education*, 79-95.