Generalisation, Assimilation, and Accommodation

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This paper builds theory by connecting Piaget's assimilation and accommodation constructs to Harel and Tall's (1991) framework for generalisation in advanced mathematics. Based on what they imagined to be the cognitive processes underlying generalisation, Harel and Tall proposed that generalisation might be expansive *(occurring when a student expands the applicability range of an existing schema without reconstructing it),* reconstructive *(occurring when a student reconstructs a schema to widen its range of applicability), or* disjunctive *(occurring when a student constructs a new, disjoint schema to deal with a new context). I contend that expansive and reconstructive generalisation align with assimilation and accommodation, respectively. I provide 'proof of concept' using data from a study of students' generalisation of graphing from* R^2 *to* R^3*. Further, I show how linking Piagetian constructs to Harel and Tall's work provides a theoretical explanation for other empirical findings about generalisation.*

Key words: generalisation, multivariable function, graphing, assimilation, accommodation

Introduction

Generalisation is a key component of mathematics. Mathematicians seek general formulae; kindergarteners generalise when they seek the next shape in a pattern; and multivariable calculus students generalise their notion of function to include functions of more than one variable. Because generalisation is so critical to mathematical thought, research that investigates how people generalise supports student learning. Moreover, students often struggle to form correct generalisations (e.g. Dorko & Weber, 2014; Jones & Dorko, 2015; Kabael, 2011; Martínez-Planell & Gaisman, 2013, 2012; Martínez-Planell & Trigueros, 2012). Generalising is important to many science, technology, engineering, and mathematics (STEM) courses. For example, students must be able to generalise their mathematics knowledge to chemistry, physics, and upper division mathematics. Difficulty generalising may contribute to students switching out of STEM studies. Efforts to better understand how students generalise and how instructors can support their generalisations could help solve the problem of retaining STEM majors (c.f. Bressoud, Carlson, Mesa, & Rasmussen, 2013; Rasmussen & Ellis, 2013; Uysal, Ellis, & Rasmussen, 2013).

Descriptions of how people generalise often come in the form of frameworks. Frameworks provide language to describe and account for qualitative differences in students' thinking and activity. Knowing what students attend to when generalising can inform instruction and the development of mathematical activities to support productive generalisation. In this paper, I connect Harel and Tall's (1991) framework for describing the "different qualities of generalisation in advanced mathematics" (p. 1) to Piaget's assimilation and accommodation constructs. This came about from my use of Harel and Tall's (1991) framework to classify empirical data, during which I often struggled to distinguish between the expansive and reconstructive generalisation categories. While the definitions of the categories seem clear, I struggled to operationalize them so they could be applied to my data. Because the descriptions of these categories seemed similar to the definitions of assimilation and accommodation (respectively), I wondered if there existed connections between the framework and the Piagetian constructs.

There are two reasons it felt worthwhile to tease apart any possible connections. The first is that such an investigation could provide insight into the cognitive processes involved in generalisation. Harel and Tall (1991) propose three ways students might generalise, but do not explain why a student might engage in one type of generalisation instead of another. Thinking about generalisation in terms of assimilation and accommodation could provide a tenable explanation. A second reason is to situate the framework in widely-understood language. This is useful because while many researchers cite Harel and Tall's (1991) definition of generalisation (e.g. Ellis, 2007; Mitchelmore, 2002) or offer hypothetical examples of Harel and Tall's three categories (e.g. Greer & Harel, 1998; Mitchelmore, 2002), the framework has been used in only three empirical studies (Fisher, 2008; Jones & Dorko, 2015; Zazkis & Liljedahl, 2002). My experience is that Harel and Tall's framework provides a powerful way to think about generalisation, but that it can be difficult to distinguish between the expansive and reconstructive categories. This difficulty may explain the lack of empirical use. I thought that if the expansive and reconstructive categories could be linked to assimilation and accommodation (respectively), it might be easier to use the framework for classifying empirical data. Moreover, the ability to talk about generalisation in terms of assimilation and accommodation affords communication about theory and results in terms of a widely-understood learning theory. This has been the case for research that has connected assimilation and accommodation to transfer (Wagner, 2010) and backward transfer (Hohensee, 2014).

This paper is structured as follows. First, I present Harel and Tall's (1991) framework and connect it to Piaget's assimilation and accommodation constructs. Then, I describe the data set and methods I used to tease apart possible connections. I follow this with an example of the utility of these connections in the context of a student generalising her thinking about graphing from R^2 to R^3. Then, I discuss how assimilation and accommodation explain other researchers' empirical findings about students generalising graphing and their notion of function from the single- to multivariable case. Finally, I offer suggestions for further research and for instruction.

Harel and Tall's (1991) Framework, Assimilation, and Accommodation

Harel and Tall (1991) proposed that generalisation in advanced mathematics fell into three categories, termed expansive, reconstructive, and disjunctive generalisation. Table 1 (next page) provides definitions of these categories and an example in the context of vector addition.

I argue that when students engage in expansive and reconstructive generalisation, they do so via assimilation and accommodation (respectively). Piaget proposed assimilation and accommodation as the mechanisms by which people learn. He discussed them in the context of schemes, or "organi[s]ation[s] of mental and affective activity" (Thompson, 2016, p. 436). *Assimilation* is defined as "the integration of new objects or new situations and events into previous schemes" (Piaget, 1980, p. 164 as cited in Steffe, 1991, p. 192). Assimilation "comes about when a cogni[s]ing organism fits an experience into a conceptual structure it already has" (von Glasersfeld, 1995, p. 62). In contrast, *accommodation* is a modification of a scheme. Accommodation occurs when a person's attempt to assimilate a situation to a scheme has an unexpected result, causing a perturbation and disequilibrium. To re-attain equilibrium, the person modifies the scheme (accommodation). In the next section, I describe the data set I used to explore the framework and Piagetian constructs.

Data Set and Methods

The data excerpts in this paper come from a longitudinal study of calculus students' generalisation of function from single- to multivariable settings (AUTHOR, 2017). I conducted

Table 1. Harel and Tall's (1991) Framework

Category	Example[1]
Expansive generalisation "occurs when the subject expands the applicability range of an existing schema without reconstructing it" (Harel & Tall, 1991, p.1). The original schema is "included directly as [a] special case in the final schema" (ibid, p.1).	A student understands vector addition <a,b> + <c,d> as performing addition twice. The student generalises her understanding of addition in R by "repeating it across more terms" (Jones & Dorko, 2015, p.156).
Reconstructive generalisation "occurs when the subject reconstructs an existing schema to widen its applicability range" (Harel & Tall, 1991, p.1). The original schema "is changed and enriched before being encompassed in the more general schema" (ibid, p.1).	A student understands vector addition <a,b> + <c,d> as performing addition twice. The student generalises her understanding of addition in R by repeating it across more terms. The student learns the geometric interpretation of vector addition as placing vectors head to tail and finding the resultant vector. The idea of vector addition "may not exist for the student in basic addition in R, and consequently the underlying idea of "addition" itself is reconstructed for the new R^2 context" (Jones & Dorko, 2015, p. 156).
Disjunctive generalisation "occurs when, on moving from a familiar context to a new one, the subject constructs a new, disjoint schema to deal with the new context and adds it to the array of schemas available" (Harel & Tall, 1991, p.1).	A student understands vector addition <a,b> + <c,d> as completely separate from addition in R. Jones and Dorko (2015) describe this hypothetical student thinking as "we might still use the same word 'addition,' but it is not the same thing as 'regular' number addition" (p. 156).

four task-based clinical interviews (Hunting, 1997) with each of five students over the span of their differential, integral, and multivariable calculus courses. The total interview time ranged from 4.25 to 5.67 hours per student. Students answered questions about single- and multivariable topics while they were enrolled in differential and (later) multivariable calculus. This design provided insight into both students' initial sense-making of how ideas from R^2 might generalise to R^3, and the sense students made of those ideas after instruction. Space constraints prohibit listing all the tasks students answered; this paper focuses on a student's response to the tasks (1) Graph $y = x$ in R^3; (2) Graph $y = 2x + 1$ in R^3; and (3) Graph $z = 4$ in R^3. Students answered these questions at the beginning of their multivariable calculus course before instruction about graphing in R^3. This timing provided insight regarding students' initial generalisations of how to graph equations. Prior to their answering the tasks, I provided students with a blank copy of R^3 axes and explained the axes' positions.

The first step in my data analysis was to review the data and identify instances of generalisation. I followed Harel and Tall's (1991) definition of generalisation as "the process of applying a given argument in a broader context" (p. 1). I then attempted to code these instances

[1] Harel and Tall (1991) offer two hypothetical examples of their categories, a detailed example of generalising how to solve systems of equations and a brief example of generalising addition in R to vector addition. Jones and Dorko (2015) offer a more detailed description of the vector addition generalisation.

as expansive generalisation, reconstructive generalisation, or disjunctive generalisation based on the definitions from Harel and Tall's (1991) framework (Table 1). I did not find any instances of disjunctive generalisation, and as such, any connections between Piagetian constructs and disjunctive generalisation are not discussed here; this is an area for future work. Finally, I took each instance and sought to code it as assimilation or accommodation based on the definitions provided above.

I gave the data presented in this paper to another researcher, who at the time was studying generalisation in real analysis from a Piagetian perspective and hence was knowledgeable about and experienced with identifying assimilation and accommodation in practise. This person coded the data separately and their codes were the same as my own. In the next section, I provide an example that illustrates the connections between assimilation and expansive generalisation and accommodation and reconstructive generalisation.

An Example: Line or Plane?

Based on my analysis, I concluded that students may engage in expansive generalisation when they assimilate a new context to an existing scheme, and they may engage in reconstructive generalisation when a new context triggers a perturbation that causes them to modify a scheme. In the following example, Wendy (pseudonym) first draws $y = x$ and $y = 2x + 1$ in R^3 as lines and $z = 4$ in R^3 as a plane. I argue that these were expansive generalisations, occurring as a result of assimilating $y = x$ and $y = 2x + 1$ in R^3 to a scheme for graphing linear functions in R^2 with $m \neq 0$ and assimilating $z = 4$ to a scheme for graphing linear function in R^2 with $m = 0$ (that is, a scheme for $y = b$). After drawing the three graphs, Wendy said she found it "interesting" that she had drawn both lines and planes. She compared the graphs and equations and reasoned that all three should be planes. That is, she engaged in reconstructive generalisation. I contend that Wendy's initial observation that she had drawn lines and a plane served as perturbation, which caused her to accommodate her scheme for graphing $y = \text{m}x + \text{b}$ equations ($m \neq 0$) in R^3.

Assimilation and Expansive Generalisation

Excerpts 1 and 2 below provide what I take as evidence of Wendy's assimilating $y = x$ and $y = 2x + 1$ in R^3 to a scheme for $y = mx + b$ ($m \neq 0$) in R^2. "Int." is short for "interviewer."

Excerpt 1. Assimilating $y = x$ in R^3 to a scheme for graphing linear functions ($m \neq 0$) in R^2

Wendy: So if you just plug in values for x and then pull out values for y, you're gonna get like 0, 0, 1, 1, 2, 2 [plots these on the xy plane as she says them] and then it's just going to continue being a straight line like this… you could choose any x value, really. I chose like 1. So if x is 1, then y is equal to x, so that's also 1.

Interviewer: Can you label some of the coordinates that you plotted?

Wendy: Okay, so this is going to be like 1, 1, 0 and then 2, 2, 0.

Interviewer: Why do we get a line here?

Wendy: The way I think of it is it's just like having a 2D graph and plotting $y = x$ and that'll give you a line, you're just taking it and adding and then ignoring the z component… if $y = x,$ you can just always assume that z is 0.Excerpt 2. Assimilating $y = 2x + 1$ in R^3 to a scheme for graphing linear functions ($m \neq 0$) in R^2

Wendy: I'm thinking that it will be like the same kind of concept where we're just ignoring z so you can say like $+0z$ here and that will give you the same equation [writes $y = 2x + 1 + 0z$]. So if you went $2x + 1$ that would be 0, 1 and then 1, 3… basically you would just take the same line that you would have with your x and y.

Interviewer: And do we get a line there?

Wendy: Yeah, that's a line… like I said we're ignoring the z component, but you can think of it as there, you're just, have it, 0 set to it.

I argue that Wendy assimilated these equations to a scheme for graphing in R^2. Wendy talked about the coordinate points as (x, y) tuples (e.g., "0, 0, 1, 1, 2, 2") as she was plotting the points (Excerpt 1). Though she described the points as (x, y, z) tuples when asked to identify points, I posit that her thinking of the points as (x, y) tuples during the act of graphing indicates that she had assimilated the question about creating a graph in R^3 to a schema for graphing in R^2. My inference is supported by Wendy's explicit statement that she saw $y = x$ in R^3 as "just like having a 2D graph and plotting $y = x$".

I contend Wendy's treatment of z facilitated her assimilation. We know that Wendy considered z because she said in both graph tasks that she was "ignoring z" (Excerpts 1 and 2) or setting it to 0 (Excerpt 3)[2]. Further, when asked what points she had plotted on her $y = x$ graph, Wendy gave (x, y, z) tuples. However, Wendy's statements about z provide evidence that she (a) explicitly considered z and (b) treated it in a way that allowed her to assimilate the $y = x$ and $y = 2x + 1$ in R^3 tasks to a scheme from R^2. This is in accordance with assimilation as "reduc[ing] new experiences to already existing sensorimotor or conceptual structures" (von Glasersfeld, 1995, p. 63). The result of Wendy's assimilation and expansive generalisation was that she drew these graphs as lines.

I argue that Wendy assimilated $z = 4$ to a different scheme. Specifically, Wendy appeared to have a scheme for $y = b$ in R^2. She drew $z = 4$ as a plane (Excerpt 3).

Excerpt 3. Assimilating $z = 4$ in R^3 to a scheme for graphing $y = b$ in R^2

Wendy: I'm thinking that whenever, no matter what x and y equal, z is always going to equal 4. So you get a plane here at 4. That's a really bad drawing of it, but, no matter what these [gestures to x axis] equal, you're always just going to get 4.

Interviewer: Can you tell me a little bit more about the 'no matter what these equal'?

Wendy: So if you're graphing, so $z = 4$, it's like saying $y = 4$ on a normal graph you get a line at y, or 4. You just get that [sketches $y = 4$ in R^2]. Because it doesn't matter what x equals. So here I'm kind of thinking that it's the same concept, that no matter what y or x equals, z is always going to equal 4.

Interviewer: Do you, as you graphed that $z = 4$, so you pretty immediately said oh, this is a plane. Did you think about this y and x graph? [points to Wendy's graph of $y = 4$ in R^2].

Wendy: I basically, I took the concept of it and applied it.

Interviewer: And what's the concept of it?

Wendy: Yeah, the concept of it is like I said even though there's no x in this equation, like we always know that y is going to be equal to 4 so it really doesn't matter what x is, so that's why there's no x in the equation.

Interviewer: How come $z = 4$ isn't just a line?

Wendy: Because you're in 3D, so if say like x was 1 and y was 2, you're always, z is going to equal 4.

I take Wendy's comment "it's the same concept" as evidence that she assimilated $z = 4$ to an already-existing scheme. What Wendy appeared to see as the "same concept" was that y equaled 4 in R^2 "[no] matter what x equals", so z would equal 4 in R^3 "no matter what y or x equals." Wendy argued that $z = 4$ was a plane using the example of (1, 2, 4) as a point on the graph.

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[2] It is important to note that for Wendy, writing $+0z$ meant that she was setting z to 0 (Excerpt 3). This contrasts a normative interpretation of $y = 2x + 1 + 0z$, in which one sees z as varying.

I contend Wendy assimilated $z = 4$ to a different scheme than the scheme to which she assimilated $y = x$ and $y = 2x + 1$. That is, Wendy appeared to have a scheme for constant functions in R^2, and an element of which was x as free. She expanded this to the R^3 case by viewing x and y as free. In contrast, she appeared to have a scheme for non-constant linear functions, an element of which was that such functions' graphs are lines. Wendy expansively generalised this scheme to the R^3 case by choosing to "ignore z" or, equivalent in her mind, 'set it to 0'.

Accommodation and Reconstructive Generalisation

The result of Wendy's assimilations to two different schemes resulted in two different graphs, triggering a perturbation that subsequently caused Wendy to reconstruct her scheme for non-constant linear equations in R^3 (Excerpt 4).

Excerpt 4. Perturbation

Wendy: It's interesting to me… That I think of that [$z = 4$] like that, and then the other ones [$y = x$ and $y = 2x + 1$] I don't think of like that. So if I, if I applied what I did in [$z = 4$] to [$y = x$ and $y = 2x + 1$] I would get planes again, which would look like this… because y is going to equal x. I feel like I'm confusing myself.

Wendy then compared her work on the three graphs, which led her to reconstruct her notion of a free variable (Excerpt 5).

Excerpt 5. Accommodation

Interviewer: Okay, so do you want to look at these again? [puts $y = x$ and $y = 2x + 1$ graphs in front of Wendy]

Wendy: So if I think about it like this [points to $z = 4$ graph], so if I thought of this [$z = 4$] like I think of this [points to $y = 2x + 1$], then this [$z = 4$] would just be a point.

Interviewer: Can you say that sentence [again]… the word 'this' gets hard when I do the audio, when I transcribe it.

Wendy: Okay so on the previous ones I was thinking of, I was thinking of this [$y = x$] as – this – the $y = x$ as just like $y = x$ and then I was thinking of it as $+0z$. And so out of that you get a line. But instead of thinking of this $+0y + 0x$, I thought of it as more of the $y = 4$. That no matter what the, no matter what the y and x values are here, the z is always going to equal 4… so if I, if I applied what I did in [$z = 4$] to [$y = x$ and $y = 2x + 1$] I would get planes again, which would look like this… because y is going to equal x. I feel like I'm confusing myself.

Interviewer: So, so do you think $y = x$ in R^3 is a plane or a line?

Wendy: My initial thought was that it was a line, but now I'm unsure… my initial thought process of it's a line is because I was thinking that you didn't change this x and y coordinate, you just laid it flat, and that is the only thing you did to make it 3D here. And so you could just graph it in 2D and then just lay it flat and put a z axis in it and that wouldn't change the $y = x$. But that was if I was thinking $+0z$ which there isn't a $+0z$. So I think that no matter what z is, y is always going to equal x. So whatever x and y are, you're going to have that plane.

I interpret the change in Wendy's graph from a line to a plane as occurring as a result of the following cognitive acts. Wendy's statement that she found it "interesting" that she had drawn a line for two of the graphs and a plane for the third suggests that she expected the graphs to look similar. The unexpected results (the graphs did not look similar) caused a perturbation. Wendy sought to re-equilibrate by comparing how she approached the $y =$ ... equations and the $z = 4$ equation. In doing so, she noticed that in the $y =$ … equations she had assumed $z = 0$, while in the z = 4 equation she had assumed x and y could take on any value. Wendy accommodated her scheme for $y = x$ in R^3 as meaning z equaled 0 meaning $z \in R$. Her initial (expansive) generalisation had been that these were lines. When she realised they were similar to $z = 4$ in that they had a free variable, she engaged in reconstructive generalisation because she widened the applicability range of her notion of free variables. That is, she applied her argument about free variables to the $y = x$ and $y = 2x + 1$ context. This allowed Wendy to "change and enrich" (Harel & Tall, 1991, p.1) her graphing schema for R^3.

Discussion

It appears that assimilation and accommodation explain a variety of empirical findings about what students generalise from R^2 to R^3. For example, researchers have observed student difficulties with graphing in R^3 (Dorko & Lockwood, 2016; Martínez-Planell & Trigueros, 2012; Trigueros & Martínez-Planell, 2010). One finding is that students may draw $f(x, y) = x^2 + y^2$ as a cylinder or a sphere because they are accustomed to $x^2 + y^2$ representing a circle in R^2 (Martínez-Planell & Gaisman, 2013). I posit students assimilate the $f(x, y) = x^2 + y^2$ to a scheme for circles in R^2, causing them to draw "circle-like" shapes in R^3. In support of this, Moore, Liss, Silverman, Paoletti, LaForest, and Musgrave (2013) have documented that students often create graphs based on *shape thinking*, or "conceiving of graphs as pictoral objects" (p. 441). That is, a possible explanation for students' graphing difficulties in R^3 is that they assimilate $f(x, y)$ equations to their schemes for the shapes of graphs in R^2, which allows them to expansively generalise by creating similar shapes on R^3 axes. As an example in a different context, researchers have found the function machine model to support students correctly generalising their notion of function from the single- to multivariable case (Dorko & Weber, 2014; Kabael, 2011). I contend that such a model allows students to assimilate multivariable functions to their function machine scheme for single-variable functions, and as such, expansively generalise their notion of function.

These examples illustrate the explanatory power of thinking about generalisation in terms of assimilation and accommodation. More broadly, they demonstrate how identifying connections between frameworks can help researchers better understand phenomena of interest. There are many generalisation frameworks, and one area for future research is to tease out relationships among them and their links to underlying theory.

Finally, Harel and Tall's stated aim in developing their framework was to "suggest pedagogical principles designed to assist students' comprehension of advanced mathematical concepts" (p. 1). One pedagogical suggestion stemming from linking the framework to Piagetian constructs is related to students' tendency to overgeneralize, such as Wendy's initial thought that the $y = \dots$ equations would be lines in R^3 as they are in R^2. When instructors notice students overgeneralising, they might consider if students are assimilating when they should be accommodating. Instructors can then help students discern features of the new context that will result in the student becoming perturbed, leading to accommodation.

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